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Topological Invariants of the Maximal Ideal Space of A Banach Algebra*

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The theme of this paper is as old as the concept of Banach algebra. Given the functor that assigns to a commutative Banach algebra A its maximal ideal space Δ_A, the urge to interpret topological data concerning Δ_A in terms of the algebraic structure of A is irresistable. The first result in this direction is due to Shilov [29]; it says that each open-compact subset of Δ_A is the support of the Gelfand transform of a unique idempotent in A. A corollary is that Δ_A itself is compact if and only if A has an identity.

The Shilov idempotent theorem can be viewed as a characterization of the zero-dimensional Cech cohomology group $H^0(\Delta_A, Z)$. In fact, it implies immediately that $H^0(\Delta_A, Z)$ is isomorphic to the additive subgroup of A generated by the idempotents in A.

An early result of Brushlinsky [9] points out that if X is compact, then the first Cech group $H^1(X, Z)$ can be identified with $C(X)^{-1}/\exp(C(X))$, where for any commutative Banach algebra A with identity, A^{-1} denotes the group of invertible elements of A and $\exp(A)$ denotes the subgroup consisting of elements of the form e^a for $a \in A$. Arens [1] and Royden [26] proved that the analogous result holds in general. That is, there is a natural isomorphism

$$H^1(\Delta_A, Z) \simeq A^{-1}/\exp(A)$$

for each commutative Banach algebra A with identity.

A few years ago, it seemed that the Shilov and Arens–Royden theorems were steps 0 and 1 in a program that would lead to characterization of

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the Cech groups $H^p(\Delta_A, Z)$ for all p. However, the next result along these lines departed from this program and introduced instead a whole series of rather mysterious (to most analysts) new invariants. This was the Arens theorem [2], which asserts that the Gelfand transform induces an isomorphism

$$[A_n^{-1}] \to [C(\Delta_A)_n^{-1}],$$

where A_n^{-1} is the invertible group in the $n \times n$ matrix algebra A_n over A and $[A_n^{-1}]$ is the factor group modulo the identity component of this group.

The significance of the Arens invariants was pointed out by Eidlin [14] and Novodvorskii [24] at about the same time. For any compact Hausdorff space X, the group $[C(X)_n^{-1}]$ is just the group $[X, GL_n(\mathbb{C})]$ of homotopy classes of maps of X into the general linear group $GL_n(\mathbb{C})$. On passing to the limit over n (using the canonical inclusion $GL_n(\mathbb{C}) \to GL_{n+1}(\mathbb{C})$), one obtains the group $K^{-1}(X)$ of Atiyah–Hirzebruch K-theory (cf., [4, 5]). Thus, if we set $K_{-1}(A) = \lim[A_n^{-1}]$, then the Arens theorem implies that there is an isomorphism

$$K_{-1}(A) \to K^{-1}(\Delta_A).$$

The group $K^{-1}(X)$ is related to the direct sum of the odd-dimensional Cech cohomology groups of X in a complicated way (cf., [13]). However, it is not always possible to determine one in terms of the other (cf., [32, Sect. 2]).

Novodvorskii [24] pointed out that one also can characterize the group $K^0(\Delta_A)$ of Atiyah–Hirzebruch K-theory in terms of the structure of A. In fact,

$$K_0(A) \simeq K^0(\Delta_A),$$

where $K_0(A)$ is the Grothendieck group of algebraic K-theory for A (cf., [6]). That is, $K_0(A)$ is the universal group generated by the semigroup (under $\oplus$) of isomorphism classes of finitely generated projective A-modules.

Although the Shilov and Arens–Royden theorems have been of great importance in Banach algebra theory and its applications, the K-theory results of Arens–Novodvorskii–Eidlin seem to have gone largely unnoticed until recently. However, the subject was revived by recent papers of Forster [15] and Lin [23] and a striking application due to Sibony and Wermer [30]. There is also a set of lecture notes (written from a fairly naive point of view) by the author [32].

A notable new result in the Forster paper is a characterization of the second Cech group $H^2(\Delta_A, Z)$. In fact,

$$\mathrm{Pic}(A) \simeq H^2(\Delta_A, Z),$$

where Pic(A) is the Picard group of A (cf., 7.7). That is, Pic(A) is the group (under $\otimes_A$) of isomorphism classes of projective A-modules M for which there is a module N with $M \otimes_A N \simeq A$.

As far as we know, the program of characterizing the Cech groups $H^p(\Delta_A, Z)$ has not progressed beyond $p = 2$. In fact, there are indications that for $p > 2$, this problem is of a strikingly different nature than the others discussed here, and may require entirely new methods (cf., 7.11).

The results outlined above and other results we shall discuss have one thing in common: They are proved using fundamental work of Grauert [16] on analytic fiber bundles (or, in the case of the Shilov and Arens–Royden theorems, using elementary facts from several complex variables that can be viewed as trivial special cases of Grauert's results). The relevant implication of Grauert's work was pointed out by Ramspott [25]: If X is a Stein space and F is a complex homogeneous space, then the inclusion map induces a bijection between homotopy classes of holomorphic maps from X to F and homotopy classes of continuous maps from X to F. Novodvorskii [24] pointed out that this could be used along with the holomorphic functional calculus for Banach algebras to relate a wide class of topological functors of the form $X \to [X, F]$ (F a complex homogeneous space) to the structure of Banach algebras. With some sharpening of Novodvorskii's results, each of the theorems mentioned above can be obtained by this technique.

In this paper, we shall state a significantly strengthened version of Novodvorskii's theorem and use it to identify a wide variety of topological invariants with Banach algebra invariants that are more or less algebraic in nature. In particular, we shall prove the theorems referred to above. These can be thought of as theorems relating the algebraic and topological versions of functors that arise in complex K-theory. We shall also derive the analogous results for real and symplectic K-theory. It is this aspect of the paper that is new (as far as we know).

The theorem from which all these results derive (the extended Novodvorskii theorem) is proved in Section 2 using Ramspott's theorem and the functional calculus. The proof is preceded by a discussion of an infinite-dimensional coordinate free version of the functional calculus that is similar to that of Craw [11]. We also give an outline of a "bundle" version of Novodvorskii's theorem. This is essentially Lin's theorem [23].

The main body of the paper (Sections 3–7) is a discussion of applications of the extended Novodvorskii theorem. It is possible to give a unified treatment of a wide variety of applications by dealing with complex homogeneous spaces that arise in connection with a finite-dimensional complex algebra Λ. The invertible group Λ^{-1} of Λ is such a space (in fact, is a complex Lie group). The space of all idempotents in Λ is a discrete union of complex homogeneous spaces, as is the space of elements in Λ with a left inverse relative to a given idempotent. Novodvorskii's theorem then yields relationships between homotopy classes of maps from Δ_A into each of these spaces and structural properties of the Banach algebra $A \otimes \Lambda$. A central example is the case where Λ is the $n \times n$ complex matrix algebra C_n and $A \otimes \Lambda$ is the $n \times n$ matrix algebra A_n over A. The results of Arens and Novodvorskii on complex K-theory arise from this example. One can achieve greater generality by allowing Λ to have a linear (not conjugate linear) involution and also studying the associated orthogonal group and space of symmetric idempotents. This provides the key to relating real and symplectic K-theory to the structure of A.

The discussion of homogeneous spaces related to Λ and the corresponding applications of Novodvorskii's theorem is carried out in Section 3. In Sections 4 and 5, we digress in order to discuss specific examples. Section 4 is devoted to showing that if Λ is the complexification of a real algebra Λ^r, then each of the associated complex homogeneous spaces has a corresponding real form of the same homotopy type (results of this sort are standard, but probably worth including). In Section 5, we specialize to the cases where Λ is the complexification of a matrix algebra over the reals, complexes, or quaternions. The results of Sections 3 and 4, when applied in these cases, yield relations between the structure of A and $[\Delta_A, F]$ when F is a classical Lie group, Grassman manifold, or Stieffel manifold associated with the reals, complexes, or quaternions. These are the standard classifying spaces that arise in the study of real, complex, or symplectic vector bundles.

In Section 6, we return to general algebras Λ and relate the results of Section 3 to the study of projective modules over $A \otimes \Lambda$. In the case where Λ has an involution, we also discuss modules with a nonsingular symmetric form. Many of the ideas here are derived from unitary K-theory a la [7].

Finally, in Section 7, we show that several functors of algebraic K-theory for an algebra $A \otimes \Lambda$ are topological invariants of Δ_A. We then specialize Λ again to obtain equivalences between algebraic and topological K-theory invariants in the real, complex, and symplectic cases. We end with a brief discussion of the status of the problem of relating the Cech cohomology groups of Δ_A to the structure of A.

Although there are probably several technically new results in these pages, the Main Theorem, its most significant applications, and most of the methods, exist in some approximation in the literature. Thus, the paper is primarily expository and is designed to provide a unified treatment of what we feel is an important branch of Banach algebra theory that deserves further attention.

The paper is aimed at the specialist in commutative Banach algebra theory who has had some exposure to algebraic topology and several complex variables.

1. Preliminaries

We shall be concerned with the category $\mathscr{A}$ of unital commutative Banach algebras and continuous unital homomorphisms. A unital algebra is an algebra with identity, while a unital homomorphism is one that preserves identities.

Throughout the paper, A will denote an object in $\mathscr{A}$ and Δ_A will denote its maximal ideal space. Thus, Δ_A is a compact Hausdorff space and the Gelfand transform $a \to \hat{a}$: $A \to C(\Delta_A)$ is a morphism in $\mathscr{A}$.

The central question of the paper: "What does topological information about Δ_A say about the structure of A?" needs to be made rather more specific. We do this as follows: The correspondence $A \to \Delta_A$, which assigns to an algebra its maximal ideal space, is a contravariant functor from $\mathscr{A}$ to the category $\mathscr{C}$ of compact Hausdorff spaces and continuous maps. We shall be interested in finding functors $F\colon \mathscr{A} \to \mathscr{D}$ (for some third category $\mathscr{D}$) that factor through $A \to \Delta_A$, that is, functors F for which there is a functor $G\colon \mathscr{C} \to \mathscr{D}$ such that $F(A)$ and $G(\Delta_A)$ are naturally isomorphic for each $A \in \mathscr{A}$. It is easy to see that such a G exists if and only if the map $F(A) \to F(C(\Delta_A))$, induced by the Gelfand transform, is an isomorphism for each $A \in \mathscr{A}$.

This program is most satisfying when F is a functor that arises naturally in the study of Banach algebras, but that is hard to compute directly, while G is a functor that can be computed using topological methods. The Arens–Royden theorem is an excellent example of this situation. Here $F(A) = A^{-1}/\exp(A)$ and $G(X) = H^1(X, Z)$.

1.1. *Homotopy*

The topological functors we shall be dealing with all have the form π^Y for some topological space Y. Here

$$\pi^Y(X) = [X, Y]$$

is the set of homotopy classes of continuous maps from X to Y. A contravariant functor G from compact spaces to sets that is naturally equivalent to π^Y will be called a homotopy functor with classifying space Y (cf., [31, Chap. 7, Sect. 7]).

Note that a homotopy functor G has the property that if f and g are homotopic maps from X_1 to X_2, then the induced maps from $G(X_2)$ to $G(X_1)$ are equal.

The classifying space for a homotopy functor is not unique. In fact, if Y is a classifying space, then Y' is also, whenever Y and Y' have the same homotopy type. Spaces Y and Y' have the same homotopy type if there are continuous maps $f\colon Y \to Y'$ and $g\colon Y' \to Y$ such that each of $f \circ g$ and $g \circ f$ is homotopic ot the identity. For example, if $Y \subset Y'$ and Y is a deformation retract of Y', then Y and Y' have the same homotopy type.

If F is a covariant functor from unital commutative Banach algebras to sets and F is naturally equivalent to the composition of $A \to \Delta_A$ with π^Y, then we shall call F a homotopy functor with classifying space Y. Our task in the ensuing section will be to determine several classes of homotopy functors on Banach algebras and to find the corresponding classifying spaces.

1.2. *H-spaces*

Although, as defined above, the functor π^Y has its range in the category of sets, quite often it can be regarded as a functor with range in a category with more structure. For example, if Y is a topological group, then $[X, Y]$ inherits a natural group structure defined from pointwise multiplication of functions from X to Y. However, there are less restrictive situations in which $[X, Y]$ inherits a group structure or other algebraic structure.

An H-space is a topological space Y with a binary operation $m\colon Y \times Y \to Y$ that has a homotopy identity, that is, a point $p \in Y$ such that the maps $y \to m(p, y)$ and $y \to m(y, p)$ are each homotopic to the identity map on Y (cf., [31, Chap. 1, Sect. 5]). If the map m is homotopy associative ($(x, y, z) \to m(m(x, y), z)$ and $(x, y, z) \to m(x, m(y, z))$ are homotopic maps), then Y is called an H-semigroup. If there is a map $j\colon Y \to Y$ such that $y \to m(y, j(y))$ and $y \to (j(y), m(y))$ are each homotopic to the constant map $y \to p$, then Y is called an H-group. Similarly, one can define H-semiring and H-ring.

If Y is an H-space then clearly, $[X, Y]$ inherits a natural operation for which the class of the constant map (with value p) is an identity. If Y is an H-semigroup or H-group, then $[X, Y]$ is a semigroup or a group for each X and π^Y may be considered a functor from compact spaces to semigroups or groups.

If Y is a space with a distinguished point p, then ΩY denotes the space of loops in Y, that is, the space of continuous maps $f\colon [0, 1] \to Y$ with $f(0) = f(1) = p$. With the compact-open topology, ΩY is an H-group (cf., [31, Chap. 1, Sect. 5]). The second loop space $\Omega^2 Y = \Omega(\Omega Y)$ is an abelian H-group.

1.3. *Inductive Topologies*

We shall have occasion to study functors π^Y, where Y is the union of an increasing family of subsets Y_i. If a set $U \subset Y$ is open if and only if for each i, $U \cap Y_i$ is open in the relative topology for Y_i, then we say that Y has the inductive topology and write $Y = \text{inj lim } Y_i$.

Let $\{Y_i\}$ be a sequence of topological spaces and suppose each Y_i is enbedded homeomorphically as a subset of Y_{i+1}. If $Y_i = \bigcup Y_i$, then Y can be topologized in such a way that $Y = \lim_i Y_i$. In fact, we simply declare a subset U to be open if $U \cap Y_i$ is open in Y_i for each i.

If X is compact and $Y = \text{inj lim } Y_i$ as above, then

$$[X, Y] = \lim[X, Y_i].$$

The proof of this fact is quite simple. The main step is to show that each compact subset K of Y must be contained in some Y_i. However, if this were not true, we could choose a sequence $\{y_n\} \subset K$ that was eventually outside each Y_i. Any set of points in such a sequence would be closed in Y since its intersection with each Y_i would be finite and hence, closed. However, this would imply that $\{y_n\}$ had no limit points, violating the assumption that K is compact.

1.4. *Bundles*

A bundle is a triple (E, X, π), where E and X are topological spaces and $\pi: E \to X$ is a continuous map. The spaces E and X are called the total space and base space of the bundle, respectively. If $x \in X$, then $\pi^{-1}(x)$ is called the fiber over x.

The trivial bundle with fiber F is the bundle $(X \times F, X, \pi)$ with $\pi: X \times F \to X$ the natural projection. A bundle (E, X, π) is said to be locally trivial with fiber F if for each $x \in X$, there is a neighborhood U of x and a homeomorphism $\varphi: \pi^{-1}(U) \to U \times F$ that is fiber preserving ($\varphi(\pi^{-1}(x)) = \{x\} \times F$ for each $x \in X$).

A section of a bundle (E, X, π) over $Y \subset X$ is a continuous map $f: Y \to E$ such that $\pi \circ f$ is the identity. If $\xi = (E, X, \pi)$, then $\Gamma(Y, \xi)$ will denote the set of all sections of ξ over Y.

Note that a section over Y of the trivial bundle $X \times F$ must have the form $x \to (x, f(x))$, where f is a continuous map of Y to F. Thus, $\Gamma(Y, \xi)$ may be identified with the space $C(Y, F)$ if ξ is the trivial bundle with fiber F.

For more on bundles see Husemöller [21].

1.5. *Extensions of Functions and Sections*

A metric space F is called an ANR (absolute neighborhood retract) if each homeomorphic image of F in a metric space X is a retract of some open set in X. This is equivalent to the property that, whenever X is a metric space and Y a closed subspace, every continuous function from Y to F has an extension to a continuous function from a neighborhood of Y to F. A convex subset of a locally convex topological vector space is an ANR. Furthermore, it is well known that if F has a countable covering by open sets which are ANR's, then F is an ANR. In particular, manifolds are ANR's (cf., [20, Chap. 1, exercises]).

We shall need bundle versions of some of the above facts.

LEMMA (cf., [23, Lemma 4]). *Let X be a separable locally compact space and let Y be a closed subspace. Let $\xi = (E, X, \pi)$ be a locally trivial bundle with fiber F that is an* ANR. *Then, each section f of ξ over Y extends to a section over a neighborhood of Y.*

Proof. We first note that the lemma holds if Y is compact and ξ is a trivial bundle. In fact, sections then can be thought of as continuous F-valued functions. Let $f: Y \to F$ be such a function. Note that F can be embedded in a Banach space B (cf., [20, Chap. 1, exercises]) and there must be a neighborhood U of F in B and a retract γ: $U \to B$ since F is an ANR. The function f extends to a continuous function from X to B. This necessarily maps some neighborhood V of Y into U. Hence, its composition with γ maps V into F and yields the required extension.

In the general case, we cover X by a countable collection of open sets V_i such that $\bar{V}_i$ is compact and ξ is trivial over a neighborhood of $\bar{V}_i$. Suppose that for some n and some compact set $K \subset X$, we have $\bigcup_{i=1}^{n} (\bar{V}_i \cap Y) \subset \operatorname{int} K$ and f extends to a section $\tilde{f}$ over $K \cup Y$. Let U_1 be a neighborhood of $\bar{V}_{n+1}$ over which ξ is trivial. Then, the restriction of $\tilde{f}$ to $(K \cup Y) \cap U_1$ extends to a neighborhood U of this set, since ξ is trivial over U_1. If W is a neighborhood of $(K \cup Y) \cap \bar{V}_{n+1}$ with compact closure in U, then $K \cup \bar{W}$ is a compact set containing $\bigcup^{n+1} \bar{V}_i$ in its interior and $\tilde{f}$ extends to $K \cup \bar{W}$. The lemma follows by induction on n.

1.6. Proposition. *Let $\xi_1 = (E_1, X, \pi_1)$ and $\xi_2 = (E_2, X, \pi_2)$ be locally trivial bundles with ξ_1 a closed sub-bundle of ξ_2. Suppose E_2 is locally compact and separable and the fiber F of ξ_1 is an* ANR. *Then, there is a neighborhood U of E_1 in E_2 and a fiber preserving retract r: $U \to E_1$ (that is $\pi_1 \circ r = \pi_2$ on U).*

Proof. We construct a third bundle (E, E_2, π) with base space E_2 and fiber F by setting $E = \{(e_1, e_2) \in E_1 \times E_2 : \pi_1(e_1) = \pi_2(e_2)\}$ and letting π: $E \to E_2$ be the natural projection. This is also a locally trivial bundle.

Since ξ_1 is a sub-bundle of ξ_2, the map $e_1 \to (e_1, e_1)$: $E_1 \to E$ defines a section of ξ over E_1. By the previous lemma, this extends to a section f over a neighborhood U of E_1 in E_2. Then, f has the form $f(e_2) = (r(e_2), e_2)$, where r is a retract of U onto E_1 such that $\pi_1 \circ g = \pi_2$.

2. The Main Theorem

The holomorphic functional calculus for Banach algebras provides a bridge between complex analysis and Banach algebra theory. This often means that a theorem in complex analysis leads to an analogous theorem in Banach algebras. Here, we are concerned with an outstanding example of this situation. Novodvorskii [24] used some deep results of Grauert [16] on holomorphic fiber bundles to prove that a wide class of functors on Banach algebras are homotopy functors (specifically, he used a slight improvement of one of Grauert's results due to Ramspott [25]). Nodvorskii's theorem exploits only the trivial bundle case of Ramspott's theorem. A recent theorem due to Lin [23] exploits the general case.

In this section, we shall present what is essentially Novodvorskii's theorem, but with some unnecessary restrictions removed. The remaining sections are devoted to exploring the implications of this theorem in special cases.

We shall also present in this section a sketch of the proof of a modified version of Lin's theorem. However, the remainder of the paper is independent of this result.

Our version of Novodvorskii's and Lin's theorems follows quite easily from Ramspott's theorem if one is willing to use the appropriate version of the holomorphic functional calculus. This is a coordinate free version of the Shilov–Arens–Calderon–Waelbroeck theorem similar to that discussed in [11]. A description of it occupies the initial portion of the section.

2.1. *Functions on an Infinite-Dimensional Space*

Let A be a Banach space and let A^* denote its dual with the weak-* topology. For $a \in A$ and $x \in A^*$ we set $\hat{a}(x) = x(a)$. Then, the map $a \to \hat{a}$ is a linear isomorphism of A onto the dual of A^*.

We denote by $\mathscr{L}(A^*)$ the collection of closed linear subspaces of finite codimension in A^*. These are exactly the kernels of linear maps of the form $\hat{\alpha}$: $A^* \to C^n$, where $\hat{\alpha} = (\hat{a}_1, \ldots, \hat{a}_n)$ for $\alpha = (a_1, \ldots, a_n) \in A^n$. If $L \in \mathscr{L}(A^*)$ we let φ_L: $A^* \to A^*/L$ be the quotient map. A subset $U \subset A^*$ will be called an L-set if $U = \varphi_L^{-1}(U_L)$, where $U_L = \varphi_L(U) \subset A^*/L$, that is, if U contains each translate of L that meets U. If U is an L-set, then an L-function on U is a complex valued function that is constant on translates of L that meet U.

If $U \subset A^*$ is open, then $C(U)$ will denote the algebra of continuous complex valued functions on U, and $\mathscr{O}(U)$ will denote the subalgebra of $C(U)$ consisting of functions that have a complex Frechet derivative at each point of U. If U is an L-set for some $L \in \mathscr{L}(A^*)$, then $C_L(U)$ and $\mathscr{O}_L(U)$ will denote the algebras of functions in $C(U)$ and $\mathscr{O}(U)$, respectively, that are L-functions. Note that $f \to f \circ \varphi_L$ is an isomorphism from $C(U_L)$ (resp. $\mathscr{O}(U_L)$) onto $C_L(U)$ (resp. $\mathscr{O}_L(U)$). That is, $C_L(U)$ (resp. $\mathscr{O}_L(U)$) is canonically isomorphic to the algebra of continuous (resp. holomorphic) functions on a domain in a finite-dimensional complex vector space. In particular, this means we can topologize $C_L(U)$

and $\mathcal{O}_L(U)$ by pulling back the compact-open topologies on $C(U_L)$ and $\mathcal{O}(U_L)$, respectively.

If, $L, K \in \mathscr{L}(A^*)$ then $L \cap K \in \mathscr{L}(A^*)$. Furthermore, if U is an L-set and V is a K-set then $U \cap V$ and $U \cup V$ are both $(L \cap K)$-sets. Because of the way the weak-* topology is defined, the open L-sets containing a point $x \in A^*$ form a neighborhood base for the topology at x as L ranges over $\mathscr{L}(A^*)$. By the obvious covering argument, if $\Delta \subset U \subset A^*$ with Δ compact and U open, then there exists $L \in \mathscr{L}(A^*)$ and an open L-set V such that $\Delta \subset V \subset U$.

PROPOSITION. *Let $\Delta \subset A^*$ be compact. Then*:

(a) *The set $\{f|_\Delta : f \in C_L(U), L \in \mathscr{L}(A^*), U$ an open L-set$\}$ is dense in $C(\Delta)$.*

(b) *If $U \supset \Delta$ is open and $f \in \mathcal{O}(U)$, then there exists $L \in \mathscr{L}(A^*)$ and an open L-set V with $\Delta \subset V \subset U$ and $f|_V \in \mathcal{O}_L(V)$.*

Proof. Part (a) follows immediately from the Stone–Weierstrass theorem. To prove part (b), note that by shrinking U if necessary, we may assume that f is bounded on U. Then, if $L \in \mathscr{L}(A^*)$ and V is an open L-set with $\Delta \subset V \subset U$, Liouville's theorem implies that $f \in \mathcal{O}_L(V)$.

2.2. *Polynomial Convexity*

We denote by $P(A^*)$ the algebra (under pointwise operations) generated by the collection of linear functions $\{\hat{a}: a \in A\}$. An element of $P(A^*)$ will be called a polynomial. The subalgebra of $P(A^*)$ consisting of elements that are constant on translates of L (for $L \in \mathscr{L}(A^*)$) will be denoted $P_L(A^*)$. Note that $P_L(A^*)$ is generated by $L^\perp = \{a \in A: \hat{a}(L) = 0\}$, which is a finite-dimensional space. Fixing a basis $a_1, \ldots, a_n$ for $L^\perp$ determines an isomorphism $p \to p \circ \hat{\alpha}$ from the ordinary polynomial algebra $P(\mathbb{C}^n)$ onto $P_L(A^*)$.

If $\Delta \subset A^*$ is compact, then its polynomial hull is the set

$$\tilde{\Delta} = \{y \in A^*: |p(y)| \leq \sup_{x \in \Delta} |p(x)|, \text{ for all } p \in P(A^*)\}.$$

We say Δ is polynomially convex if $\Delta = \tilde{\Delta}$. A polynomial polyhedron is a set of the form $V = \{y \in A^*: |p_i(y)| < 1, i = 1, \ldots, n\}$ for $p_1, \ldots, p_n \in P(A^*)$. Note that we can always choose an $L \in \mathscr{L}(A^*)$ such that $p_1, \ldots, p_n \in P_L(A^*)$ and in this case, V is an L-set. Furthermore, its image $V_L \subset A^*/L$ is an ordinary polynomial polyhedron in the finite-dimensional space A^*/L. In this case, we call V an L-polynomial polyhedron.

PROPOSITION. (a) *If* $V \subset A^*$ *is an L-polynomial polyhedron, then* $P_L(A^*)$ *is dense in* $\mathcal{O}_L(V)$;

(b) (cf., [28, 1.1]) *if* $\Delta \subset A^*$ *is compact and polynomially convex, and* $U \supset \Delta$ *is open, then there exists a polynomial polyhedron* V *with* $\Delta \subset V \subset U$.

Proof. The isomorphism $f \to f \circ \varphi_L: \mathcal{O}(V_L) \to \mathcal{O}_L(V)$ maps $P(A^*/L)$ onto $P_L(A^*)$. Since V_L is a polynomial polyhedron, part (a) follows from Runge's theorem (cf., [17, I.F.]).

Now, suppose Δ is polynomially convex and contained in the open set U. By Proposition 2.1, we may assume as well that U is a K-set for some $K \in \mathscr{L}(A^*)$. Since Δ is polynomially convex, we have that $\Delta = \bigcap \tilde{\Delta}_L$ for $L \in \mathscr{L}(A^*)$, where $\tilde{\Delta}^L$ is the $P_L(A^*)$-convex hull of Δ, that is,

$$\tilde{\Delta}^L = \{y \in A^*: |p(y)| \leqslant \sup_{x \in \Delta} |p(x)|, \text{ for } p \in P_L(A^*)\}.$$

To establish part (b), it suffices to show that $\tilde{\Delta}^L \subset U$ for some $L \in \mathscr{L}(A^*)$. In fact, in this case, we have $\tilde{\Delta}_L{}^L = \varphi_L(\tilde{\Delta}_L)$ is compact and polynomially convex in the finite-dimensional space A^*/L and contained in the open set $U_L = \varphi_L(U)$. By the usual argument (cf., [19, 2.7.4]) there is a polynomial polyhedron $W \subset A^*/L$ with $\tilde{\Delta}_L{}^L \subset W \subset U$. Then, $\tilde{\Delta}^L \subset \varphi^{-1}(W) \subset U$ (since we may assume that $L \subset K$ and, hence, that U is an L-set) and $V = \varphi^{-1}(W)$ is the required polynomial polyhedron.

We complete the argument by proving that $\tilde{\Delta}^L/U = \varnothing$ for some $L \in \mathscr{L}(A^*)$. Since Δ is compact, it is contained in a closed norm ball B of some radius. It follows from the Hahn–Banach theorem that $\tilde{\Delta}^L = \tilde{\Delta}^L + L \subset B + L$. If $L \subset K$, then $U + L = U$. Hence, if $\tilde{\Delta}^L/U \neq (\varnothing)$, then $(\tilde{\Delta}^L/U) \cap B \neq \varnothing$. However, the latter set is compact since it is weak-* closed and bounded in A^*. The collection of such sets is directed downward and has empty intersection since $\Delta = \bigcap \tilde{\Delta}^L$. Hence, $\tilde{\Delta}^L/U = \varnothing$ for some L and the proof is complete.

Part (b) of the above proposition plays the role of the Arens–Calderon lemma (cf., [3, Theorem 2.3]) in our version of the functional calculus.

2.3. *The Algebra* $\mathcal{O}(\Delta)$

If $\Delta \subset A^*$ is compact, then we define $\mathcal{O}(\Delta)$ to be the algebra of functions holomorphic in a neighborhood of Δ. That is, the collection of algebras $\{\mathcal{O}(U): \Delta \subset U\}$ is a directed system, where $f \to f|_V: \mathcal{O}(U) \to \mathcal{O}(V)$ is the bonding map for $\Delta \subset V \subset U$. The algebra $\mathcal{O}(\Delta)$ is then inj $\lim\{\mathcal{O}(U): \Delta \subset U\}$.

We can also describe $\mathcal{O}(\Delta)$ as $\operatorname{inj\,lim}\{\mathcal{O}_L(U): \Delta \subset U,\ U$ an L-set $L \in \mathcal{L}(A^*)\}$. Here, the algebras $\mathcal{O}_L(U)$ are indexed by the directed set $\mathcal{D} = \{(U, L): \Delta \subset U,\ U$ an L-set, $L \in \mathcal{L}(A^*)\}$, where $(U, L) < (V, K)$ if $V \subset U$ and $K \subset L$. The bonding map $\mathcal{O}_L(U) \to \mathcal{O}_K(V)$ is defined by $f \to f|_V$ for $(U, L) < (V, K)$. The fact that $\mathcal{O}(\Delta) = \operatorname{inj\,lim}\{\mathcal{O}_L(U): (U, L) \in \mathcal{D}\}$ follows from Proposition 2.1(a).

We topologize $\mathcal{O}(\Delta)$ by giving it the strongest locally convex topology for which each map $\mathcal{O}_L(U) \to \mathcal{O}(\Delta)$ $((U, L) \in \mathcal{D})$ is continuous. Here, $\mathcal{O}_L(U)$ has the Frechet space topology induced by the isomorphism $\mathcal{O}_L(U) \to \mathcal{O}(U_L)$. Clearly, $\mathcal{O}_L(U) \to \mathcal{O}_K(V)$ is continuous if $(L, U) < (K, V)$.

If Δ is polynomially convex, then (b) of Proposition 2.2 implies that $\{(U, L) \in \mathcal{D}:\ U$ is a polynomial polyhedron$\}$ is cofinal in $\mathcal{D}$. This, together with part (a) of Proposition 2.2 implies

PROPOSITION. *The algebra $P(A^*)$ is dense in $\mathcal{O}(\Delta)$.*

2.4. *The Functional Calculus*

At this point, we assume that A is a unital commutative Banach algebra and that $\Delta \subset A^*$ is its space of multiplicative linear functionals.

Note that Δ is polynomially convex and compact in A^*. In fact,

$$\Delta = \{x \in A^*: \hat{a}(x)\,\hat{b}(x) - (ab)\hat{}(x) = 0, \text{ for all } a, b \in A\}.$$

Since $p(x) = \hat{a}(x)\,\hat{b}(x) - (ab)\hat{}(x)$ defines a polynomial $p \in P(A^*)$, Δ is polynomially convex.

If T denotes the inverse of the map $a \to \hat{a}$ from A to the space of all weak-* continuous linear functionals on A, then T extends uniquely to an algebra homomorphism $T: P(A^*) \to A$. The version of the Shilov–Arens–Calderon–Waelbroeck theorem we shall use in the following (cf., [11]):

THEOREM. *The map $T: P(A^*) \to A$ extends to a unique continuous algebra homomorphism $T: \mathcal{O}(\Delta) \to A$.*

Proof. Since $P(A^*)$ is dense in $\mathcal{O}(\Delta)$, if T extends to a continuous map it does so uniquely. To show that it does extend, it suffices to prove that T extends from $P_L(A^*)$ to $\mathcal{O}_L(U)$ whenever $L \in \mathcal{L}(A^*)$, and U is an L-polynomial polyhedron containing Δ. (This follows from the fact that algebras $\mathcal{O}_L(U)$ of this form are cofinal in the system

$\{\mathcal{O}_L(U): (L, U) \in \mathscr{D}\}$.) However, $f \to f \circ \varphi_L: \mathcal{O}(U_L) \to \mathcal{O}_L(U)$ is an isomorphism and homeomorphism that carries $P(A^*/L)$ onto $P_L(A^*)$. Furthermore, U_L is a polynomial polyhedron in A^*/L containing $\Delta_L = \varphi_L(\Delta)$. If we impose a coordinate system in A^*/L and identify it with $\mathbb{C}^n$, then φ_L has the form $\hat{\alpha}: A^* \to \mathbb{C}^n$ for some tuple $\alpha = (a_1, ..., a_n) \in A^n$, and Δ_L is the joint spectrum of α. Hence, that T extends from $P_L(A^*)$ to $\mathcal{O}_L(U)$ follows from the ordinary functional calculus for polynomial polyhedra, which has a relatively elementary proof (cf., [3]).

If $\alpha = (a_1, ..., a_n) \in A^n$ and $\sigma(\alpha) \subset \mathbb{C}^n$ is the joint spectrum of α, then each $f \in \mathcal{O}(\sigma(\alpha))$ yields a function $f \circ \hat{\alpha} \in \mathcal{O}(\Delta)$. Hence, $T(f \circ \hat{\alpha})$ is defined. We denote this element of A by $f(\alpha)$.

Suppose $f_1, ..., f_n \in \mathcal{O}(U)$ for some open set $U \supset \Delta$, $f = (f_1, ..., f_n)$: $U \to \mathbb{C}^n$, and $g \in \mathcal{O}(f(\Delta))$. Then ,$g \circ f \in \mathcal{O}(\Delta)$, and $T(g \circ f) \in A$ is defined. We also have elements $b_i = T(f_i) \in A$ for $i = 1, ..., n$. If $\hat{\beta} = (\hat{b}_1, ..., \hat{b}_n)$, then $\hat{\beta}: A^* \to \mathbb{C}^n$ and $f: A^* \to \mathbb{C}^n$ agree on Δ. Hence, $g \circ \hat{\beta} \in \mathcal{O}(\Delta)$, and $g(\beta) = T(g \circ \hat{\beta})$ is defined. The composition property for the functional calculus can be stated as follows:

$$T(g \circ f) = g(\beta) = g(T(f)).$$

Since $T(\hat{b}_i) = b_i$, this is just the statement that T is a homomorphism if g is a polynomial. For $g \in \mathcal{O}(f(\Delta))$, it follows from continuity and Runge's theorem, provided $f(\Delta) = \hat{\beta}(\Delta)$ is polynomially convex. The general case follows from this and an appropriate use of the Arens–Calderon lemma (cf., [3, Theorem 2.3]).

2.5. *Complex Homogeneous Spaces*

A complex homogeneous space is a complex analytic manifold of the form G/H, where G is a complex Lie group, H is a closed subgroup, and G/H is the space of left cosets of G.

Suppose a complex Lie group G acts as a holomorphic transformation group on a complex analytic manifold F. If the action is transitive and $x_0 \in F$, then the map $g \to gx_0: G \to F$ induces a biholomorphic mapping $G/H \to F$, where $H = \{g \in G: gx_0 = x_0\}$ is the isotropy group of x_0 (cf., [18, Chap. II, Sect. 4]). Hence, F is a complex homogeneous space in this case. In particular, given a holomorphic representation of G on a finite-dimensional vector space, each closed orbit of G is a complex homogeneous space.

Using Grauert's machinery, Ramspott [25] proved that if $B \to X$ is a holomorphic fiber bundle over a Stein space X with fiber a complex homogeneous space G/H and structure group G, then the inclusion determines a bijection from the set of homotopy classes of holomorphic sections of $B \to X$ to the set of homotopy classes of continuous sections of $B \to X$. In the case of the trivial bundle with fiber F, this says that the inclusion determines a bijection

$$[X, F]_a \to [X, F],$$

where $[X, F]_a$ is the set of homotopy classes of holomorphic maps $f: X \to F$ and $[X, F]$ is the set of homotopy classes of continuous maps $f: X \to F$. We extend this to an infinite-dimensional version.

Let A be a Banach space and A^* its dual with the weak-* topology. If Δ, $U \subset A^*$ with U open, then $C(\Delta, F)$ ($\mathcal{O}(U, F)$) will denote the space of continuous (holomorphic) functions from Δ (U) to the complex homogeneous space F. Similarly, $C_L(\Delta, F)$ ($\mathcal{O}_L(U, F)$) will denote the set of functions in $C(\Delta, F)$ ($\mathcal{O}(U, F)$) constant on translates of $L \in \mathscr{L}(A^*)$.

2.6. Proposition. *If F is a complex homogeneous space, and $\Delta \subset A^*$ is compact and polynomially convex, then*:

(a) *If $f \in C(\Delta, F)$, then f is homotopic in $C(\Delta, F)$ to a function $h|_\Delta$ with $h \in \mathcal{O}(U, F)$ for some open set $U \supset \Delta$.*

(b) *If $U \supset \Delta$ is open, and $f, g \in \mathcal{O}(U, F)$ are such that $f|_\Delta$ and $g|_\Delta$ are homotopic in $C(\Delta, F)$, then there is an open set V with $\Delta \subset V \subset U$ and $f|_V$ and $g|_V$ homotopic in $\mathcal{O}(V, F)$.*

Proof. Part (a) will follow from Ramspott's theorem if we can prove that

(c) *There is an $L \in \mathscr{L}(A^*)$ and an L-polynomial polyhedron $U \supset \Delta$, such that f is homotopic to $g|_\Delta$ for some $g \in C_L(U, F)$.*

In fact, since $C_L(U, F)$ and $\mathcal{O}_L(U, F)$ are isomorphic to $C(U_L, F)$ and $\mathcal{O}(U_L, F)$, respectively, and since U_L is a polynomial polyhedron in A^*/L, Ramspott's theorem implies that g is homotopic to some $h \in \mathcal{O}_L(U, F) \subset \mathcal{O}(U, F)$. Then, f is homotopic to $h|_\Delta$.

Similarly, part (b) will follow directly from Ramspott's theorem if we can establish:

(d) *If $f, g \in \mathcal{O}(U, F)$ and $f|_\Delta$ is homotopic to $g|_\Delta$, then for some $L \in \mathscr{L}(A^*)$ and some L-polynomial polyhedron V with $\Delta \subset V \subset U$, $f|_V$ and $g|_V$ are homotopic in $C_L(V, F)$.*

To prove (c) and (d), we assume that F is embedded as a retract of an open set W in a Banach space B. That such an embedding is possible follows from the fact that F is an ANR (cf., 1.5).

Thus, let $r: W \to F$ (W open in B) be a retract. If $f \in C(\Delta, F)$, then by Proposition 2.1(a) there exists $L \in \mathscr{L}(A^*)$ and $g \in C_L(U, B)$ for some open L-set $U \supset \Delta$ such that $g\,|_\Delta$ is sufficiently close to f that the line segment $tf + (1-t)g\,|_\Delta$ ($t \in [0, 1]$) consists entirely of functions with values in W. Then, $t \to r \circ (tf + (1-t)g\,|_\Delta)$ yields a homotopy of f to $g\,|_\Delta$ in $C(\Delta, F)$. This establishes (c).

To prove (d), we suppose $f\,|_\Delta$ and $g\,|_\Delta$ are joined by an arc in $C(\Delta, F)$. Let $f\,|_\Delta = k_0, k_1, \ldots, k_m = g\,|_\Delta$ be points along this arc such that k_i and k_{i+1} are sufficiently close that the straight line segment joining them lies in $C(\Delta, W)$. Again, using Proposition 2.1(a), we can find $L_1 \in \mathscr{L}(A^*)$, an open L_1-set V_1 with $\Delta \subset V_1 \subset U$, and functions $f\,|_{V_1} = h_0, h_1, \ldots, h_m = g\,|_{V_1}$ in $C_{L_1}(V_1)$ such that $h_{i|\Delta}$ and k_i are sufficiently close, so that the line segment joining $h_{i|\Delta}$ to $h_{i+1|\Delta}$ lies in $C(\Delta, W)$ for each i. It follows that this continues to hold if we replace Δ by a sufficiently small neighborhood V of Δ with $V \subset V_1$. We may assume that V is an L-set for some $L \subset L_1$. We now have a piecewise linear arc joining $f\,|_V$ to $g\,|_V$ in $C(V, W)$. If we apply the retract r, we obtain the desired arc in $C(V, F)$. This establishes (d) and completes the proof.

2.7. *Novodvorskii's Theorem*

With the above infinite-dimensional form of Ramspott's theorem and the version of the functional calculus we have described, our improved version of Novodvorskii's theorem becomes a triviality.

Let A be a unital commutative Banach algebra with maximal ideal space $\Delta \subset A^*$. If $F \subset \mathbb{C}^n$ is a submanifold of some domain in $\mathbb{C}^n$, we set

$$A_F = \{\alpha \in A^n: \alpha = T(f), \text{ for some } f \in \mathcal{O}(\Delta, F)\}.$$

That is, $\alpha = (a_1, \ldots, a_n) \in A^n$ is in A_F if and only if there is a neighborhood $U \supset \Delta$ and $f = (f_1, \ldots, f_n) \in \mathcal{O}(U, F)$ such that $T(f_i) = a_i$ for $i = 1, \ldots, n$, where T is the functional calculus homomorphism.

Note that if $\alpha = (a_1, \ldots, a_n) \in A_F$, then $\hat{\alpha} = (\hat{a}_1, \ldots, \hat{a}_n)$ maps Δ into F and, hence, determines an element of $C(\Delta, F)$. If we give A_F the norm topology it inherits as a subset of A^n, then the map $\alpha \to \hat{\alpha}: A_F \to C(\Delta, F)$ is continuous.

For any space X, let $[X]$ denote the set of connectivity components of X. If X is locally path connected, then components and path components in X agree. In particular, $C(\Delta, F)$ is locally path connected since F is a retract of an open set in a Banach space. Hence, $[C(\Delta, F)] = [\Delta, F]$, the set of homotopy classes of maps from Δ to F.

Since $\alpha \to \hat{\alpha}: A_F \to C(\Delta, F)$ is continuous, it maps components to components and hence, induces a map $[A_F] \to [\Delta, F]$. Our version of Novodvorskii's theorem is:

THEOREM. *If F is a submanifold of a domain in $\mathbb{C}^n$ and a discrete union of complex homogeneous spaces, then A_F is locally path connected and the map $[A_F] \to [\Delta, F]$ is a bijection.*

Proof. We first observe that the conclusion of Proposition 2.6 continues to hold if F is a discrete union of complex homogeneous spaces. Thus, if $f \in C(\Delta, F)$, then by Proposition 2.6(a), f is homotopic to $h|_\Delta$ for some $h = (h_1, \ldots, h_n) \in \mathcal{O}(U, F)$ and some open set $U \supset \Delta$. If $\alpha = Th = (Th_1, \ldots, Th_n)$, then $\alpha \in A_F$, and $\hat{\alpha}$ is homotopic to f on Δ. Hence, $[A_F] \to [\Delta, F]$ is surjective.

Now, suppose $\alpha, \beta \in A_F$, say, $\alpha = Tf$ and $\beta = Tg$ for $f, g \in \mathcal{O}(U, F)$, and U an open set containing Δ. If $\hat{\alpha}|_\Delta = f|_\Delta$ and $\hat{\beta}|_\Delta = g|_\Delta$ are homotopic in $C(\Delta, F)$, then $f|_V$ and $g|_V$ are connected by an arc in $\mathcal{O}(V, F)$ for some open set V with $\Delta \subset V \subset U$ (by Proposition 2.6(b)). If we apply T to this arc, we obtain an arc in A_F connecting α to β. We draw two conclusions from this: (1) Since $C(\Delta, F)$ is locally path connected, so is A_F; (2) The map $[A_F] \to [\Delta, F]$ is injective. This completes the proof.

In [24], Novodvorskii proves essentially the same theorem. However, he assumes that F is open in $\mathbb{C}^n$ and that A is semisimple.

2.8. *Computing A_F*

Actually, the above theorem is not yet a true generalization of Novodvorskii's theorem. Our set A_F is defined in a rather indirect fashion, while Novodvorskii defines A_F to be the set of all $\alpha \in A^n$ such that the joint spectrum $\sigma(\alpha)$ lies in F. We shall show that the two definitions agree in the semisimple case and show how A_F can be computed even when A is not semisimple.

In what follows, $\mathcal{O}_z$ will denote the ring of germs of holomorphic functions at $z \in \mathbb{C}^n$. If F is a submanifold of a domain $U \subset C^n$, then $I(F)_z$ will denote the ideal in $\mathcal{O}_z$ consisting of germs of functions vanishing on F.

PROPOSITION. *Let F be a closed submanifold of a domain $U \subset \mathbb{C}^n$. Then:*

(a) *If A is semisimple, $A_F = \{\alpha \in A^n: \sigma(\alpha) \subset F\}$.*

(b) *If $\mathscr{F}$ is a subset of $\mathcal{O}(U)$ that generates $I(F)_z$ for each $z \in U$, then $A_F = \{\alpha \in A^n: \sigma(\alpha) \subset U$ and $f(\alpha) = 0$ for all $f \in \mathscr{F}\}$.*

Proof. If $\alpha \in A_F$, then $\alpha = T(g)$ for some holomorphic function mapping a neighborhood of Δ into F. It follows that $\sigma(\alpha) = g(\Delta) \subset F$, and by the composition property of the functional calculus (cf., 2.4), $f(\alpha) = T(f \circ \hat{\alpha}) = T(f \circ g) = 0$ for every $f \in \mathcal{O}(U)$ that vanishes on F. This yields one containment in each of parts (a) and (b).

Now suppose $\alpha \in A^n$ and $\sigma(\alpha) \subset F$. By the Arens–Calderon lemma, we may enlarge α to a tuple $\alpha_1 = (a_1, \dots, a_n, \dots, a_m) \in A^m$ $(m \geqslant n)$ and find a polynomial polyhedron $V \subset \mathbb{C}^m$ such that $\sigma(\alpha_1) \subset V$ and $\pi(V) \subset U$, where $\pi: C^m \to C^n$ is the projection defined by $\pi(z_1, \dots, z_m) = (z_1, \dots, z_n)$. Then, $F_1 = \pi^{-1}(F)$ is a closed submanifold of V and, hence, a Stein manifold. By the Docquier–Grauert theorem (cf., [17, VIII. C. 8.]), there is a neighborhood V_1 of F_1 and a holomorphic retract $\varphi: V_1 \to F_1$. If we set $W = \hat{\alpha}_1^{-1}(V_1)$ and $g = \pi \circ \varphi \circ \hat{\alpha}_1$ on W, then W is a neighborhood of Δ and $g: W \to F$ is a holomorphic map which agrees with $\hat{\alpha}$ on Δ. By definition, $T(g) = \pi \circ \varphi(\alpha_1) \in A_F$.

If A is semisimple, then $\alpha = T(g)$, since $\hat{\alpha} = g = T(g)\hat{}$ on Δ. Thus, $\alpha \in A_F$ and the proof is complete in case (a).

For case (b), we assume that $f(\alpha) = 0$ for all $f \in \mathscr{F}$. If we set $f_1 = f \circ \pi \in \mathcal{O}(V)$ for each $f \in \mathscr{F}$, then $f_1(\alpha_1) = f(\pi(\alpha_1)) = f(\alpha) = 0$. Furthermore, $\mathscr{F}_1 = \{f_1 : f \in \mathscr{F}\}$ generates $I(F_1)_z$ for each $z \in V$ and hence, by Cartan's theorem B, $\mathscr{F}_1$ generates the ideal in $\mathcal{O}(V)$ of functions vanishing on F_1 (cf., [17, VIII. A. 15]). Now, $\pi \circ \varphi - \pi: V \to C^n$ vanishes on F_1; thus, its coordinate functions are sums of multiples of elements of $\mathscr{F}_1$. We conclude that $T(g) - \alpha = (\pi \circ \varphi)(\alpha_1) - \pi(\alpha_1) = 0$ and $\alpha \in A_F$. This completes the proof of part (b).

Part (b) of the above proposition will allow us to effectively apply Theorem 2.7 without assuming semisimplicity.

Theorem 2.7 and the above proposition constitute our extension of Novodvorskii's theorem and provide the basis for the results of the remaining sections. We close this section with an outline of a similar extension of Lin's theorem. Although this result will possibly have many applications in Banach algebra theory, we have not attempted to include any of them here. (Lin's main application [23, Theorem 3] is obtained here in Sect. 3 as an application of Theorem 2.7.) Thus, the remainder of the paper will be independent of the discussion below.

2.9. *Complex Homogeneous Bundles*

Let F, E, and X be analytic spaces, let G be a complex Lie group, and let $\pi: E \to X$ be a holomorphic map. Suppose G acts transitively and holomorphically on F. We shall call (E, X, π) a complex homogeneous bundle (with fiber F and group G) provided:

(i) There is an open cover $\{V_i\}$ of X and, for each i, a biholomorphic map $f_i: \pi^{-1}(V_i) \to V_i \times F$ mapping $\pi^{-1}(x)$ into $\{x\} \times F$ for each $x \in V_i$.

(ii) For each pair (i, j) of indices, there is a holomorphic map $g_{ij}: V_i \cap V_j \to G$ such that $(f_i \circ f_j^{-1})(x, u) = (x, g_{ij}(x)u)$ for all $x \in V_i \cap V_j$ and $u \in F$.

The two conditions above say that (E, X, π) is a locally trivial fiber bundle with fiber F and with transition functions given by holomorphic G-valued functions (cf., [21]).

Recall that we denote the set of sections over $Y \subset X$ of a bundle $\xi = (E, X, \pi)$ by $\Gamma(Y, \xi)$. For a complex homogeneous bundle and an open set $U \subset X$, $\Gamma_a(U, \xi)$ will denote the set of sections that are holomorphic maps from U to E.

Ramspott's theorem [25] says that the inclusion $\Gamma_a(X, \xi) \to \Gamma(X, \xi)$ induces a bijection $[\Gamma_a(X, \xi)] \to [\Gamma(X, \xi)]$ of homotopy classes if $\xi = (E, X, \pi)$ is a complex homogeneous bundle and X is a Stein space.

If $\varphi: X \to Y$ is a holomorphic map of analytic spaces and $\xi = (E, Y, \pi)$ is a complex homogeneous bundle over Y, then there is an induced complex homogeneous bundle $\varphi^*\xi$ over X. This is the bundle $(\varphi^*E, X, \varphi^*\pi)$, where $\varphi^*E = \{(x, e) \in X \times E: \pi(e) = \varphi(x)\}$ and $\varphi^*\pi: \varphi^*E \to X$ is projection on the first coordinate. Note that φ also determines a map $\varphi^*: \Gamma(Y, \xi) \to \Gamma(Y, \xi) \to \Gamma(X, \varphi^*\xi)$ by $\varphi^*f(x) = (x, f(\varphi(x))) \in \varphi^*E$ for $f \in \Gamma(Y, \xi)$. If f is a holomorphic section, so is φ^*f.

2.10. *Bundles over a Neighborhood of Δ*

Now, all of the above makes perfectly good sense even if the base space X is infinite-dimensional. Specifically, we are interested in complex homogeneous bundles over a neighborhood U of Δ in A^*. Given such a bundle $\xi = (E, U, \pi)$, it is easy to see that, after possibly shrinking U, we may assume that U and the sets V_i of 2.9(i) are L-sets and the transition functions g_{ij} of 2.9(ii) are in $\mathcal{O}_L(V_i \cap V_j, G)$ for some $L \in \mathscr{L}(A^*)$ (cf., 2.1). This implies that ξ has the form $\varphi_L{}^*\xi_L$ for a complex homogeneous bundle $\xi_L = (E_L, U_L, \pi_L)$ over the domain $U_L \subset A^*/L$. Note that if this is true of one $L \in \mathscr{L}(A^*)$, it is also true of all its subspaces in $\mathscr{L}(A^*)$.

In what follows, we shall assume that $\xi = (E, U, \pi)$ is a bundle of the above form $\xi = \varphi^* \xi_L$, where $\xi_L = (E_L, U_L, \pi_L)$, E_L is a closed submanifold of an open set $V \subset U_L \times \mathbb{C}^n$, and $\pi_L: E_L \to U_L$ is the restriction of $(z, w) \to z: U_L \times \mathbb{C}^n \to U_L$ to E_L. This allows us to consider E as the subset $\{(x, w) \in U \times \mathbb{C}^n: (\varphi(x), w) \in E_L\}$ of $U \times \mathbb{C}^n$. The continuous (holomorphic) sections of ξ over $K \subset U$ then can be thought of as continuous (holomorphic) maps $f: K \to \mathbb{C}^n$ with graphs $\{(x, f(x)): x \in K\}$ contained in E.

Given a holomorphic section f over a neighborhood V with $\Delta \subset V \subset U$, the functional calculus yields an element $T(f) \in A^n$. The set of all such elements will be denoted A_ξ. If $\alpha \in A_\xi$, say $\alpha = T(f)$ for $f \in \Gamma_\alpha(U, \xi)$, then, $\hat{\alpha}$ agrees with f on Δ and hence, determines a continuous section in $\Gamma(\Delta, \xi)$.

Our version of Lin's theorem is the following:

2.11. THEOREM. *Let ξ be a complex homogeneous bundle over a neighborhood U of Δ in A^*. Assume ξ is embedded as above in the trivial bundle $(U \times \mathbb{C}^n, U, \pi)$. Then, A is locally path connected and the Gelfand transform induces a bijection $[A_\xi] \to [\Gamma(\Delta, \xi)]$.*

Outline of Proof. The proof follows the same outline as the proof of Theorem 2.7. That is, one uses Ramspott's theorem to establish a version of Proposition 2.6 for sections of ξ rather than for F-valued functions. The proof then proceeds routinely as in Theorem 2.7. The only additional difficulty encountered in modifying Proposition 2.6 is the following: Where before we embedded F as a retract of an open set in a Banach space B, we must now embed E as a fiber preserving retract of an open set W in a space $U \times B$. (That is, the retract $r: W \to E$ should map $\pi^{-1}(x)$ into itself for each $x \in U$, where $\pi: U \times B \to U$ is the natural projection). Now, we have already assumed that E is embedded in $U \times \mathbb{C}^n$. Hence, it suffices to prove that there is a fiber preserving retract of some neighborhood of E in $U \times \mathbb{C}^n$ onto E. Also, we have assumed that $\xi = \varphi_L^* \xi_L$ for some $L \in \mathscr{L}(A^*)$ and a bundle $\xi_L = (E_L, U_L, \pi_L)$ over $U_L \in A^*/L$. Hence, to obtain the required retract, it suffices to obtain E_L as a fiber preserving retract of a neighborhood in $U_L \times \mathbb{C}^n$ and then pull back along φ_L. Thus, we have reduced the problem to the finite-dimensional case. The existence of the required retract now follows from Proposition 1.6.

2.12. *Remark.* There is also an analog of Proposition 2.8 in the bundle case.

Suppose $\xi = \varphi_L{}^*\xi_L$ for $L \in \mathscr{L}(A^*)$ and $\xi_L = (E_L, U_L, \pi_L)$ a complex homogeneous bundle over $U_L \subset \Delta_L$. If we impose a coordinate system on A^*/L, then we may identify A^*/L with $\mathbb{C}^k$ for some k and $\varphi_L: A^* \to A^*/L$ with $\hat{\beta}$ for some $\beta \in A^k$.

We suppose E_L is embedded as a sub-bundle of $U_L \times \mathbb{C}^n \subset \mathbb{C}^k \times \mathbb{C}^n$ and is a closed submanifold of an open set $W \subset U_L \times \mathbb{C}^n$.

PROPOSITION. *Under the above conditions,*

(a) *If A is semisimple then $A_\xi = \{\alpha \in A^n : \sigma(\beta, \alpha) \subset E_L\}$.*

(b) *If $\mathscr{F}$ is a family of holomorphic functions on W such that $\mathscr{F}$ generates $I(E_L)_z$ for each $z \in W$, then $A_\xi = \{\alpha \in A^n : \sigma(\beta, \alpha) \in W$ and $f(\beta, \alpha) = 0$ for all $f \in \mathscr{F}\}$.*

The proof proceeds as in Proposition 2.8. The only essential difference is that here, the retract obtained from the Docquier–Grauert theorem must be chosen so that it is fiber preserving. This amounts to a routine extension of the Dorquier–Grauert theorem. We omit the details.

3. EXAMPLES INVOLVING FINITE-DIMENSIONAL ALGEBRAS

We shall be interested in applications of the results of Section 2 in the cases where F is one of several standard homogeneous spaces constructed from the classical complex Lie groups $GL_n(\mathbb{C})$, $O_n(\mathbb{C})$, and $\mathrm{Sp}_n(\mathbb{C})$. These spaces are of interest in complex, real, and symplectic K-theory.

Initially, rather than discussing the real, complex, and symplectic cases separately, we shall study a situation that has each of these as a special case. This invovles a class of spaces that arise in connection with the study of finite-dimensional algebras with involution.

We begin with some technical results regarding a Banach algebra B and then specialize to the cases where B is either a finite-dimensional algebra A, or $A \otimes A$ for a commutative Banach algebra A.

3.1. *Involutions*

Throughout this section, B will denote a unital complex Banach algebra (not necessarily commutative).

By an involution on B, we shall mean a continuous linear map $b \to b'$:

$B \to B$ such that $(bc)' = c'b'$ and $b'' = b$ for all $b, c \in B$. We emphasize that we are referring to *linear* not conjugate linear involutions.

Elements $b \in B$ such that $b' = b$ will be called symmetric, while elements b such that $b' = -b$ will be called antisymmetric. The sets of symmetric and antisymmetric elements are closed linear subspaces of B, which we denote by B^+ and B^-, respectively.

An element $b \in B^{-1}$ will be called orthogonal if $b' = b^{-1}$. The orthogonal elements form a closed subgroup of B^{-1}, which we denote by $O(B)$.

We shall call a unital Banach algebra with a specified involution a symmetric algebra.

Our discussion of symmetric algebras in this section and of symmetric modules in the next section is derived in part from unitary K-theory as discussed in [7]. However, we have taken some liberties with the terminology.

3.2. *Hyperbolic Algebras*

Given a unital Banach algebra B, there is a standard way of constructing a symmetric algebra $H(B)$ called the hyperbolic algebra of B. We first let B^{op} denote the opposite algebra of B. That is, B^{op} is B as a Banach space, but the product $a \circ b$ of two elements in B^{op} is defined to be their product ba in B in reverse order. Then, $H(B)$ is the algebra direct sum $B \oplus B^{\mathrm{op}}$ with involution defined by

$$(a, b)' = (b, a).$$

We clearly have

$$H(B)^+ = \{(b, b) \colon b \in B\},$$

$$H(B)^- = \{(b, -b) \colon b \in B\},$$

and

$$O(H(B)) = \{(b, b^{-1}) \colon b \in B^{-1}\}.$$

Note that the last of these equalities implies that B^{-1} is isomorphic as a topological group to $O(H(B))$. Thus, invertible groups in Banach algebras are special cases of orthogonal groups in symmetric algebras.

3.3. *The Exponential map*

If B is a unital Banach algebra, it is a standard fact that the exponential map $b \to e^b$ maps B into B^{-1} and is a homeomorphism of a neighborhood of 0 in B onto a neighborhood of 1 in B^{-1}. There is an obvious analog for the orthogonal group in a symmetric algebra B, which we establish below.

It follows from the continuity of involution and the power series for e^b that $(e^b)' = e^{b'}$ for $b \in B$. Hence, if $b \in B^-$ then $(e^b)' = e^{-b} = (e^b)^{-1}$. That is, the exponential map sends B^- into $O(B)$.

Let W be the neighborhood of 1 in $\mathbb{C}$ defined by

$$W = \{z \in \mathbb{C} : |1 - z| < 1 \text{ and } |1 - z^{-1}| < 1\}.$$

If $a \in B^{-1}$ and $\sigma(a) \subset V$, then $\sigma(a^{-1}) \subset V$ since $\sigma(a^{-1}) = \sigma(a)^{-1}$. Furthermore, it is easily seen that $\sigma(a') = \sigma(a)$. Hence, if ln is the principal branch of the log function on W, then $\ln(a)$, $\ln(a^{-1})$, and $\ln(a')$ are all defined by the functional calculus.

From the power series for ln on W and the continuity of involution, we conclude that $\ln(a') = \ln(a)'$. Since $\ln(z^{-1}) = -\ln(z)$ on W, we have that $\ln(a^{-1}) = -\ln(a)$. Hence, if $a \in O(B)$ $(a' = a^{-1})$ and $\sigma(a) \subset W$, then $\ln(a)' = -\ln(a)$ and $\ln(a) \in B^-$.

From the above, we conclude:

Lemma. *There are neighborhoods $U \ni 0$ in B^- and $V \ni 1$ in $O(B)$ such that $b \to e^b$ maps U homeomorphically onto V with inverse map $a \to \ln(a)$.*

The neighborhoods U and V can be taken to be

$$V = \{a \in O(B) : \sigma(a) \subset W\} \qquad \text{and} \qquad U = \{b \in B^- : e^b \in V\}.$$

Note that if we apply the lemma to a hyperbolic algebra $H(B)$, use the characterizations of $H(B)^-$ and $O(H(B))$ of 3.2, and note that $e^{(a,b)} = (e^a, e^b)$, then we simply recover the fact that $b \to e^b$ maps a neighborhood of 0 in B homeomorphically onto a neighborhood of 1 in B^{-1}.

Let $\exp(B)$ (resp. $\exp(B^-)$) denote the subgroup of B^{-1} (resp. $O(B)$) generated by elements e^b for $b \in B$ (resp. $b \in B^-$).

3.4. Proposition. *The identity components of B^{-1} and $O(B)$ are open subgroups and are* $\exp(B)$ *and* $\exp(B^-)$, *respectively.*

Proof. It is immediate from the lemma that $\exp(B)$ and $\exp(B^-)$ are open, connected subgroups of B^{-1} and $O(B)$, respectively. Since open subgroups are also closed, the proposition follows.

3.5. *Idempotents*

We denote the set of idempotents in B by $\mathrm{ID}(B)$. If B is symmetric, then $\mathrm{ID}(B^+)$ will denote the set of symmetric idempotents in B. Each of these is a topological space in the norm topology of B.

The group $O(B)$ acts on $\mathrm{ID}(B^+)$ through inner automorphism. In fact, if $p \in \mathrm{ID}(B^+)$ and $a \in O(B)$, then $apa^{-1} = apa'$ is an idempotent and $(apa')' = ap'a' = apa'$. In particular, the identity component $\exp(B^-)$ of $O(B)$ acts on $\mathrm{ID}(B^+)$.

We also have that B^{-1} and, hence, $\exp(B)$ act by inner automorphism on $\mathrm{ID}(B)$. This is a special case of the above, since the map $p \to (p, p)$: $\mathrm{ID}(B) \to \mathrm{ID}(H(B)^+)$ is a homoemorphism and the action of B^{-1} on $\mathrm{ID}(B)$ carries over (under the isomorphism $B^{-1} \to O(H(B))$) to the action of $O(H(B))$ on $\mathrm{ID}(H(B)^+)$.

PROPOSITION. *The connectivity components of* $\mathrm{ID}(B^+)$ *are relatively open and are exactly the orbits of the action of* $\exp(B^-)$ *on* $\mathrm{ID}(B^+)$. *Similarly, the components of* $\mathrm{ID}(B)$ *are relatively open and are the orbits of the action of* $\exp(B)$ *on* $\mathrm{ID}(B)$.

Proof. We prove the first statement. The second can be obtained as a special case of the first by passing to the hyperbolic algebra (although a direct proof is trivial).

Since the orbits of the $\exp(B^-)$ action are connected, the proof will be complete if we can show that they are open.

For $p, q \in \mathrm{ID}(B^+)$ we set

$$a = pq + (1 - p)(1 - q).$$

Then $pa = pq = aq$. Furthermore, if q is fixed, there is a neighborhood $V \ni q$ in $\mathrm{ID}(B^+)$ such that for $p \in V$ we have $\| a - 1 \| < 1$ and $\| a'a - 1 \| < 1$. It follows that $a \in B^{-1}$ and $\ln(a'a)$ is defined and is symmetric.

Since $p' = p$ and $q' = q$ we have

$$apa^{-1} = q = q' = (a')^{-1} pa',$$

from which it follows that $a'a$ commutes with p. Hence, $u = \ln(a'a)$ and $b = e^{(1/2)u}$ also commute with p. Note that b is symmetric.

If $c = ab^{-1}$, then $cpc^{-1} = apa^{-1} = q$. Also, $c'c = b^{-1}a'ab^{-1} = 1$ and, hence, $c \in O(B)$. Now, c depends continuously on p as p ranges over V, and $c = 1$ if $p = q$. Hence, for p in a possibly smaller neighborhood $U \ni q$, c will be in the identity component $\exp(B^-)$ of $O(B)$. This establishes that orbits of $\exp(B^-)$ acting on $\mathrm{ID}(B^+)$ are open.

3.6. *Relatively Orthogonal Elements*

Let $p \in \mathrm{ID}(B^+)$ be fixed. We denote by $O(B, p)$ the set of all elements $a \in B$ such that

$$ap = a \qquad \text{and} \qquad a'a = p.$$

This is the set of elements a for which p is a right identity and a' is a left inverse relative to p.

The group $O(B)$ acts on $O(B, p)$ by left multiplication. Hence, the identity component acts as well and has connected orbits.

PROPOSITION. *The components of $O(B, p)$ are relatively open and are the orbits of the action of* $\exp(B^-)$ *on $O(B, p)$.*

Proof. We must show that orbits are open. For $a, b \in O(B, p)$ we set

$$u = 1 - aa' + ba'$$

and note that $ua = b$.

Suppose that $bb' = aa' = q$. Then, q is a symmetric idempotent, $(1 - q)a = (1 - q)b = 0$, and $u = (1 - q) + ba'$. It follows that $u'u = uu' = 1$ and hence, that $u \in O(B)$. For b sufficiently close to a, u will be in the identity component $\exp(B^-)$. Thus, we have that b is in the same orbit as a if it is close to a and satisfies $bb' = aa'$.

Now, suppose $bb' \neq aa'$. If a is sufficiently close to b, the symmetric idempotent bb' will be in the orbit of aa' under the action of $\exp(B^-)$ by inner automorphisms (Proposition 3.5). In this case, we have $vbb'v' = aa'$ for some $v \in \exp(B^-)$. Furthermore, for b sufficiently close to a we can choose v so that vb is in a prescribed neighborhood of a. Since $(vb)(vb)' = aa'$, the argument of the previous paragraph shows that, if this neighborhood is small enough, $vb = ua$ for some $u \in \exp(B^-)$. Hence, b and a are in the same orbit and orbits are open.

As usual, by applying the above result to the hyperbolic algebra $H(B)$, we obtain a result about nonsymmetric algebras B. The reader may verify that in this case the result is the following: If $p \in \mathrm{ID}(B)$ let $T(B, p)$ denote the set of pairs $(a, b) \in B \oplus B$ such that

$$ap = a, \qquad pb = b, \qquad ba = p.$$

Let $\exp(B)$ act on $T(B, p)$ by $u(a, b) = (ua, bu^{-1})$. Then, the orbits of this action are open and coincide with the components of $T(B, p)$.

Now, if a is the first element of a pair $(a, b) \in T(B, p)$ then $\{c \in B: (a, c) \in T(B, p)\}$ is obviously a convex set. It follows easily that there is a one to one correspondence between the components of $T(B, p)$ and those of

$$L(B, p) = \{a \in B: (a, b) \in T(B, p), \text{ for some } b \in B\}$$

and that the components of this set are the orbits of the action of $\exp(B)$ on $L(B, p)$ given by left multiplication.

3.7. *Tensor Algebras*

Let Λ be a finite-dimensional unital Banach algebra, and let A be a commutative unital Banach algebra. Then, under the greatest cross norm, $A \otimes \Lambda$ is a unital Banach algebra. If Λ is symmetric, then $A \otimes \Lambda$ is also symmetric under the involution $\alpha \to \alpha'$, which is defined by $(a \otimes \lambda)' = a \otimes \lambda'$ on elementary tensors.

If we specify a vector space basis for Λ, then Λ may be identified (as a vector space) with $\mathbb{C}^n$. Relative to this same basis $A \otimes \Lambda$ may be regarded as a copy of A^n. Multiplication in Λ is then given by a quadratic function $m: \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ and multiplication in $A \otimes \Lambda$ is the corresponding map $(\alpha, \beta) \to m(\alpha, \beta): A^n \times A^n \to A^n$ given by the functional calculus. Similarly, an involution on Λ is given by a linear function from $\mathbb{C}^n$ to $\mathbb{C}^n$ and the corresponding involution on $A \otimes \Lambda$ is given by applying this to elements of A^n via the functional calculus.

The Gelfand transform $a \to \hat{a}: A \to C(\Delta_A)$ induces a homomorphism $a \otimes \lambda \to \hat{a} \otimes \lambda: A \otimes \Lambda \to C(\Delta_A) \otimes \Lambda$. We denote this map by $\alpha \to \hat{\alpha}$ and call it the Gelfand transform for $A \otimes \Lambda$. Note that $C(\Delta_A) \otimes \Lambda = C(\Delta_A, \Lambda)$, the algebra of continuous maps from Δ_A to Λ. Hence, for $\alpha \in A \otimes \Lambda$, $\hat{\alpha}$ is a Λ-valued continuous function on Δ_A.

If we choose a basis for Λ and represent $A \otimes \Lambda$ as A^n, then the Gelfand transform of $\alpha \in A \otimes \Lambda$ is obtained by taking the Gelfand transform of each entry of α regarded as an element of A^n.

As an example, consider the case where $\Lambda = \mathbb{C}_n$ is the $n \times n$ complex matrix algebra. Then, $A \otimes \Lambda = A_n$ is the $n \times n$ matrix algebra over A. Transposition defines an involution on $\mathbb{C}_n$ and the corresponding involution on A_n is also transposition. The Gelfand transform of $\alpha \in A_n$ is obtained by taking the Gelfand transform of each matrix entry.

We are now prepared to give several applications of the machinery of Section 2. In the following, Λ is symmetric whenever it needs to be for the statement to make sense.

3.8. THEOREM. *The Gelfand transform induces bijections:*

(a) $[(A \otimes \Lambda)^{-1}] \to [\Delta_A, \Lambda^{-1}]$,
(b) $[O(A \otimes \Lambda)] \to [\Delta_A, O(\Lambda)]$,
(c) $[\mathrm{ID}(A \otimes \Lambda)] \to [\Delta_A, \mathrm{ID}(\Lambda)]$,
(d) $[\mathrm{ID}((A \otimes \Lambda)^+)] \to [\Delta_A, \mathrm{ID}(\Lambda^+)]$,
(e) $[L(A \otimes \Lambda, p)] \to [\Delta_A, L(\Lambda, p)]$, *for* $p \in \mathrm{ID}(\Lambda)$,
(f) $[O(A \otimes \Lambda, p)] \to [\Delta_A, O(\Lambda, p)]$, *for* $p \in \mathrm{ID}(\Lambda^+)$.

Proof. The entire theorem will follow if we can prove (d) and (f). In fact, (b) is the special case of (f) in which $p = 1$, (a) and (c) follow from (b) and (d) applied to the hyperbolic algebra $H(\Lambda)$ (since $H(A \otimes \Lambda) = A \otimes H(\Lambda)$), and (e) follows from (f) applied to $H(\Lambda)$ and the comment following Proposition 3.6.

The group $O(\Lambda)$ is a closed analytic subgroup of Λ^{-1} and hence, a complex Lie group. Its identity component is $\exp(\Lambda^-)$. It follows from 3.5 and 3.6 that each of $\mathrm{ID}(\Lambda^+)$ and $O(\Lambda, P)$ is a closed submanifold of Λ and a discrete union of complex homogeneous spaces. Hence, if F is either of these spaces, we have $[A_F] = [\Delta_A, F]$ by Theorem 2.7. It remains to prove that A_F is the right space in each of the two cases.

In the first case, $F = \mathrm{ID}(\Lambda^+)$ is the zero set of the holomorphic map $\varphi: \Lambda \to \Lambda \times \Lambda$ defined by $\varphi(\lambda) = (\lambda^2 - \lambda, \lambda' - \lambda)$. At $p \in \mathrm{ID}(\Lambda^+)$ the complex differential $d\varphi_p$ of this map is given by

$$d\varphi_p(\lambda) = (\lambda p + p\lambda - \lambda, \lambda' - \lambda),$$

where we identify the complex tangent space to Λ at p with Λ. If the kernel of $d\varphi_p$ agrees with the tangent space of $\mathrm{ID}(\Lambda^+)$ at p, then the coordinate functions of φ necessarily generate the ideal in $\mathcal{O}p(\Lambda)$ determined by the subvariety $\mathrm{ID}(\Lambda^+)$. We shall show that this is the case.

Since $\exp(\Lambda^-)$ acts transitively on each component of $\mathrm{ID}(\Lambda^+)$, the tangent space to $\mathrm{ID}(\Lambda^+)$ at p consists of the elements

$$(d/dt)(e^{t\mu} p e^{-t\mu})\,|_{t=0} = \mu p - p\mu, \quad (\mu \in \Lambda^-).$$

If $\lambda = \mu p - p\mu$ is such an element, then

$$\lambda p + p\lambda = \mu p - p\mu p + p\mu p - p\mu = \lambda$$

and

$$\lambda' = p\mu' - \mu' p = -p\mu - \mu p = \lambda.$$

Hence, $\lambda \in \ker d\varphi_p$. Conversely, if $\lambda \in \ker d\varphi_p$ then $\lambda p + p\lambda = \lambda$ and $\lambda' = \lambda$. It follows that $2p\lambda p = p\lambda p$ and hence, that $p\lambda p = 0$. Thus, if $\mu = (1 - 2p)\lambda$ then

$$\mu p - p\mu = (1 - 2p)\,\lambda p + p\lambda = \lambda$$

and

$$\mu + \mu' = 2\lambda - 2p\lambda - 2\lambda p = 0.$$

Hence, λ is in the tangent space to $\mathrm{ID}(\Lambda^+)$.

We may now apply Proposition 2.8 in the case where $F = \mathrm{ID}(\Lambda^+)$ and $\mathscr{F}$ is the set of coordinate functions of φ. We conclude that

$$\begin{aligned} A_F &= \{p \in A \otimes \Lambda : \varphi(p) = 0\} \\ &= \{p \in A \otimes \Lambda : p^2 - p = p' - p = 0\} = \mathrm{ID}((A \otimes \Lambda)^+). \end{aligned}$$

This completes the proof of part (d).

In part (f), $F = O(\Lambda, p) = \{\lambda \in \Lambda : \varphi(\lambda) = 0\}$, where $\varphi : \Lambda \to \Lambda \times \Lambda$ is the holomorphic map defined by

$$\varphi(\lambda) = (\lambda'\lambda - p, \lambda p - \lambda).$$

The proof in this case follows as above if we can show that for each $\nu \in O(\Lambda, p)$ the kernel of $d\varphi_\nu$ is the tangent space of $O(\Lambda, p)$ at ν. Again, since $\exp(\Lambda^-)$ acts transitively on components of $O(\Lambda, p)$ we have that the tangent space to $O(\Lambda, p)$ at ν consists of those elements $\lambda \in \Lambda$ of the form

$$\lambda = (d/dt)(e^{t\mu}\nu)\,|_{t=0} = \mu\nu \quad (\mu \in \Lambda^-).$$

The differential $d\varphi_\nu$ is given by

$$d\varphi_\nu(\lambda) = (\lambda'\nu + \nu'\lambda, \lambda p - \lambda).$$

Hence, if $\lambda = \mu\nu$ with $\mu \in \Lambda^-$, then

$$\begin{aligned} \lambda'\nu + \nu'\lambda &= \nu'\mu'\nu + \nu'\mu\nu = 0, \\ \lambda p - \lambda &= \mu\nu p - \mu\nu = \mu(\nu p - \nu) = 0, \end{aligned}$$

and $\lambda \in \ker d\varphi_\nu$. Conversely, if $\lambda \in \ker d\varphi_\nu$, and we set $\mu = \lambda\nu'$, then

$$\mu' = (\lambda\nu')' = \nu\lambda' = -\lambda'\nu = -\mu$$

and

$$\lambda = \lambda p = \lambda\nu'\nu = \mu\nu.$$

Thus, $\ker d\varphi_\nu$ is the tangent space of $O(\Lambda, p)$ at ν and part (f) follows.

Note that Propositions 3.4–3.6 applied to $A \otimes \Lambda$ allow us to interpret each of the sets on the left in the above theorem as the set of orbits of an action of $\exp(A \otimes \Lambda)$ or $\exp(A \otimes \Lambda)^-$ on a subset of $A \otimes \Lambda$ that is defined in purely algebraic terms. Thus, the theorem identifies several functors on Banach algebras A that are defined in essentially algebraic terms, but which are homotopy functors with the indicated classifying spaces.

We should point out that part (a) of the theorem is a special case of a theorem of Davie [12], in which he proves the corresponding result with Λ an infinite-dimensional Banach algebra and $A \otimes \Lambda$ replaced by the completed tensor algebra $A \hat{\otimes} \Lambda$. Also, in the case where $\Lambda = \mathbb{C}_n$ and $A \otimes \Lambda = A_n$, part (a) is just the Arens theorem [2].

3.9. *Matrices over Λ*

If Λ is an algebra with involution $\lambda \to \lambda'$, then the $n \times n$ matrix algebra Λ_n is also an algebra with a natural involution defined by $(\lambda_{ij})' = (\lambda'_{ij})^t$. Thus, the preceding results apply equally well to Λ_n and $A \otimes \Lambda_n = (A \otimes \Lambda)_n$.

If p is the idempotent in Λ_n that has ones in the first k places on the diagonal and zeros elsewhere, then a matrix $\lambda \in \Lambda_n$ satisfies $\lambda p = \lambda$ if and only if the last $n - k$ column vectors of λ are zero. Thus, we may identify $\{\lambda \in \Lambda_n : \lambda p = \lambda\}$ with the space Λ_k^n of $n \times k$ matrices over Λ. If we define $\lambda \to \lambda' : \Lambda_k^n \to \Lambda_n^k$ by $(\lambda_{ij})' = (\lambda'_{ij})^t$, then $\lambda'\lambda \in \Lambda_k$ and

$$O(\Lambda_n, p) = \{\lambda \in \Lambda_k^n : \lambda'\lambda = 1 \in \Lambda_k\}.$$

We shall also denote this set by $O_k^n(\Lambda)$. If $n = k$ then this is $O(\Lambda_n)$, which we shall also denote by $O_n(\Lambda)$. Similarly, we set

$$L_k^n(\Lambda) = L(\Lambda_n, p) = \{\lambda \in \Lambda_k^n : \mu\lambda = 1, \text{ for some } \lambda \in \Lambda_n^k\}.$$

In case $k = n$, this is just Λ^{-1}, which we also denote by $GL_n(\Lambda)$. Parts (e) and (f) of Theorem 3.8 yield:

COROLLARY. *For $k \leqslant n$, the Gelfand transform induces bijections*

$$[O_k{}^n(A \otimes \Lambda)] \to [\Delta_A, O_k{}^n(\Lambda)],$$
$$[L_k{}^n(A \otimes \Lambda)] \to [\Delta_A, L_k{}^n(\Lambda)].$$

We close this section with two examples of purely algebraic results that combine portions of Theorem 3.8.

3.10. PROPOSITION. *If $p, q \in \mathrm{ID}(A \otimes \Lambda)$, then p and q are similar via an element of $(A \otimes \Lambda)^{-1}$ if and only if $\hat{p}, \hat{q} \in \mathrm{ID}(C(\Delta_A, \Lambda))$ are similar via an element of $C(\Delta_A, \Lambda)^{-1}$.*

Similarly, if $p, q \in \mathrm{ID}((A \otimes \Lambda)^+)$ then p and q are similar via an element of $O(A \otimes \Lambda)$ if and only if $\hat{p}, \hat{q} \in \mathrm{ID}(C(\Delta_A, \Lambda)^+)$ are similar via an element of $O(C(\Delta_A, \Lambda))$.

Proof. We prove the first statement. The proof of the second is entirely analogous.

If $upu^{-1} = q$ for $u \in (A \otimes \Lambda)^{-1}$, then $\hat{u}\hat{p}(\hat{u})^{-1} = \hat{q}$ for $\hat{u} \in C(\Delta_A, \Lambda)^{-1}$. The converse is the nontrivial part.

Suppose $v \in C(\Delta_A, \Lambda)^{-1}$ and $v\hat{p}v^{-1} = \hat{q}$. By Theorem 3.8(a), v is in the same component of $C(\Delta_A, \Lambda)^{-1}$ as some $\hat{w}$ with $w \in (A \otimes \Lambda)^{-1}$. Hence, if $q_1 = wpw^{-1}$, then $\hat{q} = v\hat{p}v^{-1}$ and $\hat{q}_1 = \hat{w}\hat{p}(\hat{w})^{-1}$ are in the same component of $\mathrm{ID}(C(\Delta_A, \Lambda))$. By Theorem 3.8(c), q and q_1 are in the same component of $\mathrm{ID}(A \otimes \Lambda)$. Hence, $q = uq_1u^{-1}$ for some $u \in \exp(A \otimes \Lambda)$ by Proposition 3.5. Thus,

$$q = uq_1u^{-1} = uwp(uw)^{-1}$$

and the proof is complete.

In the case where $\Lambda = \mathbb{C}_n$, the above proposition says that idempotent matrices $p, q \in A_n$ are similar if and only if $\hat{p}, \hat{q} \in C(\Delta_A)_n$ are similar. In particular, if we think of A_n as an algebra of operators on $A^n = A \oplus \cdots \oplus A$, then an idempotent $p \in A^n$ can be diagonalized by a change of (A-module) basis in A^n if and only if $\hat{p}$ can be diagonalized by a ($C(\Delta_A)$-module) change of basis in $C(\Delta_A)^n$.

3.11. PROPOSITION. *If $u, v \in L_k{}^n(A \otimes \Lambda)$ $(O_k{}^n(A \otimes \Lambda))$, then there exists $w \in GL_n(A \otimes \Lambda)$ $(O_n(A \otimes \Lambda)$ such that $u = wv$ if and only if there exists $g \in GL_n(C(\Delta_A, \Lambda)$ $(O_n(C(\Delta_A, \Lambda)))$ such that $\hat{u} = g\hat{v}$.*

Proof. Let $\hat{u} = g\hat{v}$ with $u, v \in L_k{}^n(A \otimes \Lambda)$ and $g \in GL_n(C(\Delta_A, \Lambda))$. By Theorem 3.8(a), there exists $w_1 \in GL_n(A \otimes \Lambda)$ such that g and $\hat{w}_1$

are in the same component of $GL_n(C(\Delta_A, A))$. Hence, if $u_1 = w_1 v$ then $\hat{u}_1 = \hat{w}_1 \hat{v}$ and $\hat{u} = g\hat{v}$ are in the same component of $L_k^n(C(\Delta_A, A))$ and, thus, u_1 and u are in the same component of $L_k^n(A \otimes A)$ by Corollary 3.9. By Proposition 3.6, we have $u = w_2 u_1$ for some $w_2 \in \exp((A \otimes A)_n)$. Then, $u = w_2 w_1 v$ and the proof is complete.

Suppose $u \in L_k^n(A \otimes A)$ is obtained from the square matrix $w \in GL_n(A \otimes A)$ by deleting the last $n - k$ columns. Then, $w^{-1}u = v$, where v is the $n \times k$-matrix with one's on the main diagonal and zeroes elsewhere. Conversely, with this v, if $u = wv$ for some $w \in GL_n(A \otimes A)$, then u is obtained from w by deleting the last $n - k$ columns. Combining this observation with the corresponding observation for $O_k^n(A \otimes A)$ and the above proposition, we obtain:

COROLLARY. *If $u \in L_k^n(A \otimes A)$ ($O_k^n(A \otimes A)$), then u can be filled out to an $n \times n$ matrix in $GL_n(A \otimes A)$ ($O_n(A \otimes A)$) if and only if $\hat{u}$ can be filled out to an $n \times n$ matrix in $GL_n(C(\Delta_A, A))$ ($O_n(C(\Delta_A, A))$).*

In the case where $A = \mathbb{C}$ and $u \in L_k^n(A)$, the above proposition is due to Lin [23]. Sibony and Wermer [30] have used this result, in the case where A is a certain kind of algebra of holomorphic functions, to obtain conditions on a set of elements of A that ensure that the set generates A.

4. Complexification of a Real Symmetric Algebra

In this section, we present some technical results that are preliminary to our discussion of examples in the next section. What we want to show is that for a large class of symmetric algebras A, the associated complex homogeneous spaces $O(A)$, $\mathrm{ID}(A^+)$, and $O_k^n(A)$ have "real forms" of the same homotopy type. It is this that allows us to relate topological invariants involving real classifying spaces (such as real K-theory) to the structure of a complex Banach algebra.

4.1. *Complexification*

Let A^r be a real symmetric algebra, that is, an algebra over the reals with a real linear involution $\mu \to \mu'$. We obtain a complex algebra with involution by setting $A = \mathbb{C} \otimes_R A^r$. Alternatively, one can describe A as the algebra of formal sums $\mu + i\nu$ ($\mu, \nu \in A^r$), where $(\mu_1 + i\nu_1)(\mu_2 + i\nu_2) = (\mu_1\mu_2 - \nu_1\nu_2) + i(\mu_1\nu_2 + \nu_1\mu_2)$.

Now, Λ not only has a linear involution defined by

$$(\mu + i\nu)' = \mu' + i\nu',$$

it also has a conjugate linear involution $\lambda \to \lambda^*$ defined by

$$(\mu + i\nu)^* = (\mu' - i\nu').$$

That is, $\lambda^* = (\bar{\lambda})' = (\lambda')^-$, where $(\mu + i\nu)^- = \mu - i\nu$.

We shall be interested in the case where Λ is a C^*-algebra under the involution $\lambda \to \lambda^*$. This means that Λ has a norm and $*$ preserving representation as an algebra of operators on a Hilbert space. In the cases of interest to us, Λ will be finite-dimensional and hence, the C^*-algebra condition means that it has a norm and $*$ preserving representation as a subalgebra of a complex matrix algebra.

Given a C^*-algebra Λ that possesses, in addition to the involution $\lambda \to \lambda^*$, a linear involution $\lambda \to \lambda'$, such that $(\lambda')^* = (\lambda^*)'$, there is a real linear subalgebra Λ^r such that Λ is the complexification of Λ^r as above. In fact, it suffices to set

$$\Lambda^r = \{\lambda \in \Lambda : \bar{\lambda} = \lambda\},$$

where $\bar{\lambda} = (\lambda')^*$.

In the remainder of this section, Λ will be a C^*-algebra obtained as above by complexifying a real symmetric algebra Λ^r.

4.2. *Polar Decomposition*

Since Λ has a linear involution $\lambda \to \lambda'$, we have the orthogonal group $O(\Lambda)$ defined as before. However, we now have a unitary group

$$U(\Lambda) = \{\lambda \in \Lambda^{-1} : \lambda^* = \lambda^{-1}\}$$

defined as well. Furthermore, note that

$$U(\Lambda) \cap O(\Lambda) = O(\Lambda^r),$$

where

$$O(\Lambda^r) = \{\mu \in \Lambda^r : \mu' = \mu^{-1}\},$$

and we identify Λ^r with its image $\{\lambda \in \Lambda : \lambda = \bar{\lambda}\}$ in Λ.

An element λ of Λ will be called hermitian if $\lambda^* = \lambda$.

LEMMA. (a) *Each* $\lambda \in A^{-1}$ *has a unique factorization as* $\lambda = \mu e^{\rho}$, *where* μ *is unitary and* ρ *is hermitian;*

(b) *If* $\lambda \in O(A)$ *and* $\lambda = \mu e^{\rho}$ *as above, then* $\mu \in O(A^r)$ *and* ρ *is skew-symmetric* ($\rho' = -\rho$).

(c) *The map* $\rho \to e^{\rho}$ *is a homeomorphism of the space of hermitian elements of* A *onto the space of positive definite elements of* A^{-1}.

Proof. Parts (a) and (c) are standard facts from spectral theory.

To prove part (b), we note that $e^{2\rho} = \lambda^*\lambda$ and that $\lambda^*\lambda$ is orthogonal since λ is. Thus, $(e^{2\rho})' = e^{2\rho'} = e^{-2\rho}$. Since ρ' is also hermitian, we conclude that $\rho' = -\rho$. It follows that e^{ρ} is orthogonal and hence, that $\mu = \lambda e^{-\rho}$ is also orthogonal.

4.3. PROPOSITION. *Topologically,* A^{-1} *is the cartesian product of* $U(A)$ *with a real Banach space. Similarly,* $O(A)$ *is the cartesian product of* $O(A^r)$ *with a real Banach space.*

Proof. This follows immediately from the lemma. The Banach spaces in question are the space of hermitian elements of A and the space of hermitian skew-symmetric elements of A, respectively.

4.4. *Idempotents*

As before, $\mathrm{ID}(A)$ and $\mathrm{ID}(A^+)$ are the spaces of idempotents and symmetric idempotents of A, respectively. However, now we also have the space $\mathrm{ID}(A^h)$ of hermitian idempotents. Note that

$$\mathrm{ID}(A^+) \cap \mathrm{ID}(A^h) = \mathrm{ID}((A^r)^+),$$

where the latter space is the space of symmetric idempotents in A^r.

PROPOSITION. *The space* $\mathrm{ID}(A^h)$ *is a deformation retract of* $\mathrm{ID}(A)$, *while* $\mathrm{ID}(A^r)^+)$ *is a deformation retract of* $\mathrm{ID}(A^+)$.

Proof. We prove that $\mathrm{ID}(A^r)^+)$ is a deformation retract of $\mathrm{ID}(A^+)$. The corresponding result for $\mathrm{ID}(A^h)$ and $\mathrm{ID}(A)$ follows the same pattern, but is easier.

If $p \in \mathrm{ID}(A^+)$, we set $u = p^*p + (1-p)^*(1-p)$. Then, $u' = pp^* + (1-p)(1-p)^*$, since $p' = p$ and $(p^*)' = p^*$. A simple computation shows that

$$u = \tfrac{1}{2}[(2p-1)^*(2p-1) + 1].$$

Since u is the sum of a positive definite element and a positive scalar, it is invertible. Further computation shows that

$$p^* = upu^{-1}, \qquad uu' = u'u, \qquad u'up = pu'u.$$

It follows that $u'u$ is positive definite and symmetric. Hence, $(u'u)^{1/2}$ is positive definite, symmetric, and commutes with p and u. If $v = u(u'u)^{-1/2}$, then v is positive definite,

$$v'v = vv' = 1, \qquad p^* = vpv^{-1}.$$

If we set $v = e^w$ for w hermitian and skew-symmetric and

$$q = e^{-(1/2)w}p^*e^{(1/2)w} = e^{(1/2)w}pe^{-(1/2)w},$$

then q is hermitian and symmetric ($q \in \mathrm{ID}(\Lambda^r)^+$) and

$$t \to e^{tw}pe^{-tw}, \qquad (t \in [0, \tfrac{1}{2}])$$

yields an arc connecting p to q.

Note that w was constructed in a well-defined fashion from p and clearly depends continuously on p. Furthermore, if p is already hermitian, then $u = 1$, $v = 1$, $w = 0$ and $q = p$. It follows that we have defined a deformation retract of $\mathrm{ID}(\Lambda^+)$ onto $\mathrm{ID}(\Lambda^r)^+)$.

4.5. *Nonsquare Orthogonal Matrices*

The involutions $\lambda \to \lambda'$ and $\lambda \to \lambda^*$ extend in the obvious way to matrices over Λ. That is, if $\lambda \in \Lambda_k{}^n$ is an $n \times k$ matrix over Λ, then λ' (resp. λ^*) $\in \Lambda_k{}^n$ is the result of applying $'$ (resp. $*$) to each entry of λ and then transposing. In the case of square matrices, the algebra $\Lambda_n = \Lambda_n{}^n$ is simply another C^*-algebra satisfying the hypotheses we have assumed on Λ. However, we are also interested in the nonsquare case.

As in Section 3, we set (for $n > k$)

$$O_k{}^n(\Lambda) = \{\lambda \in \Lambda_k{}^n : \lambda'\lambda = I\}.$$

We also have the real space:

$$O_k{}^n(\Lambda^r) = \{\lambda \in (\Lambda^r)_k^n : \lambda'\lambda = 1\}.$$

For $\lambda \in O_k{}^n(\Lambda)$, we have that $\lambda \in O_k{}^n(\Lambda^r)$ if and only if $\lambda' = \lambda^*$.

PROPOSITION. *The space $O_k{}^n(\Lambda^r)$ is a deformation retract of $O_k{}^n(\Lambda)$.*

Proof. If $u \in O_k{}^n(\Lambda)$, then $u'u = 1$, while $uu' = p$ is a symmetric idempotent in $\mathrm{ID}(\Lambda_n{}^+)$. By the argument in 5.4, we have that p determines a skew-symmetric hermitian element $w \in \Lambda_n$ such that

$$q = e^{(1/2)w} p e^{-(1/2)w}$$

is symmetric and hermitian. If we set

$$v = e^{(1/2)w} u,$$

then $v \in O_k{}^n(\Lambda)$ and $vv' = q$. Since v depends continuously on p and hence, on u, we conclude that there is a deformation retract of $O_k{}^n(\Lambda)$ onto its subset consisting of elements v with vv' hermitian.

Now, let v be in the latter set and note that v^*v is a positive definite element of Λ_k^{-1}. Hence, $v^*v = e^h$ for a unique hermitian matrix in Λ_k. If $y = ve^{(1/2)h}$, then $y^*y = y'y = 1$ and $yy' = vv' = q$. Hence,

$$y' = y^*yy' = y^*q = (qy)^* = y^*.$$

It follows that y is real and hence, that $y \in O_k{}^n(\Lambda^r)$.

Clearly, y depends continuously on v. We conclude that $O_k{}^n(\Lambda^r)$ is a deformation retract of $O_k{}^n(\Lambda)$.

The analogous result for the space $L_k{}^n(\Lambda)$ is somewhat easier to prove. We set

$$U_k{}^n(\Lambda) = \{\lambda \in U_k{}^n \colon \lambda^*\lambda = 1 \in U_k\}.$$

Then:

4.6. Proposition. *The space $U_k{}^n(\Lambda)$ is a deformation retract of $L_k{}^n(\Lambda)$.*

Proof. If we represent Λ as an algebra of operators on a Hilbert space H, then $u \in \Lambda_k{}^n$ may be represented as an operator from H^k to H^n. If $u \in L_k{}^n(\Lambda)$ then u has a left inverse. It follows that $u^*u \colon H^k \to H^k$ is positive definite and invertible. If $w = \ln u^*u$ then $v = ue^{-(1/2)w} \in U_k{}^n(\Lambda)$ and $t \to ue^{-tw}$ $(t \in [0, \frac{1}{2}])$ is an arc connecting u to v. Since w depends continuously on u, we have defined a deformation retract $u \to v \colon L_k{}^n(\Lambda) \to U_k{}^n(\Lambda)$.

5. Examples

The simple finite-dimensional real algebras are exactly the matrix algebras over the reals, complexes, and quaternions. In this section, we shall discuss the results of Sections 3 and 4 in the cases where A^r is one of these algebras and A is its complexification. For each of these examples, A^r has a natural involution. This and the results of Section 4 allow us to retain the special properties of A^r even though we must pass to the complex algebra A in order to apply the results of Section 3.

5.1. *The Real Case*

The $n \times n$ real matrix algebra R_n is a real symmetric algebra with transposition as the involution. The complexification of R_n is $\mathbb{C}_n$. For $\lambda \in \mathbb{C}_n$ the matrices λ', $\bar{\lambda}$, and λ^* are the ordinary transpose, conjugate, and conjugate transpose matrices, respectively. Note that $\mathbb{C}_n$ is a C^*-algebra under $\lambda \to \lambda^*$.

The algebra $A \otimes \mathbb{C}_n$ is just the $n \times n$ matrix algebra A_n over A with transposition as involution.

The groups $O(R_n)$ and $O(\mathbb{C}_n)$ are simply the real and complex orthogonal groups $O_n(R)$ and $O_n(\mathbb{C})$. Similarly, the group $O(A \otimes \mathbb{C}_n) = O(A_n)$ is just $\{\alpha \in A_n : \alpha^t = \alpha^{-1}\}$, which we call the $n \times n$ orthogonal group over A and denote by $O_n(A)$.

For $k \leqslant n$, the spaces $O_k{}^n(R)$, $O_k{}^n(\mathbb{C})$, and $O_k{}^n(A) = O_k{}^n(A \otimes \mathbb{C})$ can be described as the spaces of $n \times k$ matrices α over R, $\mathbb{C}$, and A, respectively, such that $\alpha^t\alpha = 1$. In particular, $O_k{}^n(R)$ can be described as the space of all orthonormal k-tuples (i.e., all k-frames) of vectors in R^n. This is referred to as the Stiefel manifold of k-frames in R^n by topologists (cf., [21, Chap. 7]).

The spaces $\mathrm{ID}(R_n{}^+)$, $\mathrm{ID}(\mathbb{C}_n{}^+)$, and $\mathrm{ID}(A_n{}^+)$ are, of course, simply the spaces of symmetric (under transposition) idempotent matrices in R_n, $\mathbb{C}_n$, and A_n, respectively.

Combining Theorem 3.8 with the topological results of the previous section, we have bijections

$$[O_n(A)] \to [\Delta_A, O_n(\mathbb{C})] = [\Delta_A, O_n(R)],$$

$$[O_k{}^n(A)] \to [\Delta_A, O_k{}^n(\mathbb{C})] = [\Delta_A, O_k{}^n(R)],$$

and

$$[\mathrm{ID}(A_n{}^+)] \to [\Delta_A, \mathrm{ID}(\mathbb{C}_n{}^+)] = [\Delta_A, \mathrm{ID}(R_n{}^+)].$$

Furthermore, by 3.4–3.6, we also have that the connectivity components of $O_n(A)$, $O_k{}^n(A)$, and $\mathrm{ID}(A_n{}^+)$ are open sets and are the orbits, in each case, of the appropriate action of $\exp(A_n{}^-)$.

5.2. *Real Grassman Manifolds*

Each linear subspace of R^n is the range of a unique symmetric idempotent in R_n. Hence, $\mathrm{ID}(R_n{}^+)$ may be regarded as the space of all linear subspaces of R^n. For $k \leqslant n$ the space of k-dimensional linear subspaces of R^n (equivalently, rank k symmetric idempotents) is the Grassman manifold $G_k(R^n)$ (cf., [21, Chap. 7]).

If $\mathrm{ID}(\mathbb{C}_n{}^+)$ is the subset of $\mathrm{ID}(\mathbb{C}_n{}^+)$ consisting of matrices of rank k, then the deformation retract of $\mathrm{ID}(\mathbb{C}_n{}^+)$ onto $\mathrm{ID}(R_n{}^+)$ must yield a deformation retract of $\mathrm{ID}_k(\mathbb{C}_n{}^+)$ onto $G_k(R^n)$ since rank is a continuous function on $\mathrm{ID}(\mathbb{C}_n{}^+)$. Let $\mathrm{ID}_k(A_n{}^+)$ be the set of $p \in \mathrm{ID}(A_n{}^+)$ such that $\hat{p}(x) \in \mathrm{ID}_k(\mathbb{C}_n{}^+)$ for each $x \in \Delta_A$. It is clear that if p and q are in the same component of $\mathrm{ID}(A_n{}^+)$ and $p \in \mathrm{ID}_k(A_n{}^+)$, then so is q. It follows that we have bijections

$$[\mathrm{ID}_k(A_n{}^+)] \to [\Delta_A, \mathrm{ID}_k(\mathbb{C}_n{}^+)] = [\Delta_A, G_k(R^n)],$$

for each k. Hence, the Grassman manifold $G_k(R^n)$ is a classifying space for $A \to [\mathrm{ID}_k(A_n{}^+)]$.

If a symmetric idempotent $p \in A_n$ lies in $\mathrm{ID}_k(A_n{}^+)$ we will say it has constant rank k.

5.3. *The Complex Case*

Suppose we choose for A^r a complex matrix algebra $\mathbb{C}_n$, considered as a real algebra, with conjugate transpose as involution. We will describe the complexification of this real algebra.

Consider the hyperbolic algebra $\mathbb{C} \oplus \mathbb{C}$ of $\mathbb{C}$ (cf., 3.2). Here $\mathbb{C} \oplus \mathbb{C}$ has coordinatewise operations and the complex linear involution defined by $(\lambda, \mu)' = (\mu, \lambda)$. There is also a conjugate linear involution defined by $(\lambda, \mu)^* = (\lambda^-, \bar{\mu})$. The two involutions commute and they coincide exactly on the set of elements of the form (λ, λ^-). This set is a real subalgebra which is real isomorphic to $\mathbb{C}$ under the map $\lambda \to (\lambda, \lambda^-)$: $\mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$. Note that this map is involution preserving if the involution on $\mathbb{C}$ is conjugation. It follows that $\mathbb{C} \oplus \mathbb{C}$ is the complexification of the real symmetric algebra $\mathbb{C}$.

It follows easily that the complexification of the real symmetric algebra $\mathbb{C}_n$ is $\mathbb{C}_n \oplus \mathbb{C}_n$ with involution $(\lambda, \mu) \to (\mu^t, \lambda^t)$. However, this algebra is isomorphic as a symmetric algebra to the hyperbolic algebra $H(\mathbb{C}_n) = \mathbb{C}_n \oplus \mathbb{C}_n^{\mathrm{op}}$ under the map $(\lambda, \mu) \to (\lambda, \mu^t)$. Hence, the complexification of $\mathbb{C}_n$ is just $H(\mathbb{C}_n)$.

When applied to a hyperbolic algebra, $H(\Lambda)$, the bijection (b), (d), and (f) of Theorem 3.8 simply yield the bijections (a), (c), and (d) for Λ. Hence, if we regard $\mathbb{C}_n$ as a real algebra with real involution (conjugate transpose), complexify, and apply our results on symmetric algebras, we are led straight back to considering $\mathbb{C}_n$ as a complex algebra and to ignoring the involution.

Thus, the relevant classifying spaces from Theorem 3.8 are $\mathbb{C}_n^{-1} = GL_n(\mathbb{C})$, $\mathrm{ID}(\mathbb{C}_n)$, and $L_k{}^n(\mathbb{C})$ (cf., 3.9). Each of these has a corresponding real form as a deformation retract. For $GL_n(\mathbb{C})$ the real form is $U_n(\mathbb{C}) = U(\mathbb{C}_n)$ (cf., 4.3). For $\mathrm{ID}(\mathbb{C}_n)$ it is the space $\mathrm{ID}((\mathbb{C}_n)^h)$ of hermitian idempotents (cf., 4.4). The space $L_k{}^n(\mathbb{C})$ may be regarded as the space of linearly independent k-tuples of vectors in $\mathbb{C}^n$. Its real form is the space $U_k{}^n(\mathbb{C})$ of orthonormal k-tuples in $\mathbb{C}^n$ (i.e., the space of $n \times k$ matrices u with $u^*u = 1$). This is the complex Stiefel manifold of complex orthonormal k-frames $V_k(\mathbb{C}^n)$ (cf., [21, Chap. 7]).

By Theorem 3.8 we have bijections

$$[GL_n(A)] \to [\Delta_A, GL_n(\mathbb{C})] = [\Delta_A, U_n(\mathbb{C})],$$

$$[\mathrm{ID}(A_n)] \to [\Delta_A, \mathrm{ID}(\mathbb{C}_n)] = [\Delta_A, \mathrm{ID}((\mathbb{C}_n)^h)],$$

$$[L_k{}^n(A)] \to [\Delta_A, L_k{}^n(\mathbb{C})] = [\Delta_A, U_k{}^n(\mathbb{C})],$$

where $GL_n(A) = A_n^{-1}$.

As in 5.1, we can pass to the subsets $\mathrm{ID}_k(\mathbb{C}_n) \subset \mathrm{ID}(\mathbb{C}_n)$ and $\mathrm{ID}_k((\mathbb{C}_n)^h) \subset \mathrm{ID}((\mathbb{C}_n)^h)$ consisting of matrices of rank k, and the subset $\mathrm{ID}_k(A_n) \subset \mathrm{ID}(A_n)$ consisting of matrices of constant rank k. The space $\mathrm{ID}_k((\mathbb{C}_n)^h)$ consists of the orthogonal projections of rank k and hence, may be identified with the Grassman manifold $G_k(\mathbb{C}^n)$ of all k-dimensional subspaces of $\mathbb{C}^n$. As in 5.1, we have bijections

$$[\mathrm{ID}_k(A_n)] \to [\Delta_A, \mathrm{ID}_k(\mathbb{C}_n)] = [\Delta_A, G_k(\mathbb{C}^n)]$$

for each k.

5.4. *The Symplectic Case*

The algebra H of quaternions may be regarded as a real subalgebra of $\mathbb{C}_2$, in fact, as the real linear span of the elements

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The conjugate transpose operation $\lambda \to \lambda^*$ yields an involution on H such that $\mathbf{i}^* = -\mathbf{i}$, $\mathbf{j}^* = -\mathbf{j}$, $\mathbf{h}^* = -\mathbf{k}$. This agrees with the involution $\lambda \to \lambda'$, where

$$\lambda' = \mathbf{j}\lambda^t\mathbf{j}^{-1}.$$

In fact, this latter involution defines a complex linear involution on all of $\mathbb{C}_2$ and $H = \{\lambda \in \mathbb{C}_2 : \lambda' = \lambda^*\}$. Since the involutions $\lambda \to \lambda'$ and $\lambda \to \lambda^*$ commute, we conclude that $\mathbb{C}_2$ with the involution $\lambda \to \lambda'$ is the complexification of H as a symmetric algebra.

It follows from the above that the matrix algebra H_n has $H_n^{\mathbb{C}} = \mathbb{C}_{2n}$ as complexification. Here, the involution is defined by

$$u' = J_n u^t J_n^{-1},$$

where $J_n \in \mathbb{C}_{2n}$ is the matrix which, when written in 2×2 blocks, has $\mathbf{j}$ down the diagonal and zeros elswehere. Furthermore, H_n is the real subalgebra of $\mathbb{C}_{2n}$ consisting of elements u for which $u^* = u'$.

The orthogonal groups of H_n and $H_n^{\mathbb{C}}$ relative to the involution $u \to u'$ are denoted by $\mathrm{Sp}_n(R)$ and $\mathrm{Sp}_n(\mathbb{C})$, respectively, and are called the real and complex symplectic groups.

The spaces $O_k^n(H)$ and $O_k^n(H^{\mathbb{C}})$ will be denoted $\mathrm{Sp}_k^n(R)$ and $\mathrm{Sp}_k^n(\mathbb{C})$, respectively. If we give H^n the (H-valued) inner product defined by $(\lambda_1, \ldots, \lambda_n)(\mu_1, \ldots, \mu_n) = \mu_1'\lambda_1 + \cdots + \mu_n'\lambda_n$, then $\mathrm{Sp}_k^n(R)$ can be regarded as the set of all orthonormal k-tuples of vectors in H^n. This is called the Stiefel manifold of symplectic k-frames.

We denote the spaces of idempotents in H_n and $H_n^{\mathbb{C}}$ that are symmetric relative to $u \to u'$ by $\mathrm{ID}^S(H_n)$ and $\mathrm{ID}^S(H_n^{\mathbb{C}})$, respectively. Each idempotent in $\mathrm{ID}^S(H_n)$ is determined by its range as a projection on H^n. Hence, there is a one to one correspondence between elements of $\mathrm{ID}^S(H_n)$ and H-linear subspaces of H^n.

The algebra $A \otimes H_n$ is the matrix algebra A_{2n} with involution defined by $u \to u' = J_n u^t J_n^{-1}$. We denote this symmetric algebra by H^A, its orthogonal group by $\mathrm{Sp}_n(A)$, and its space of symmetric idempotents by $\mathrm{ID}^S(H_n^A)$. The space $\mathcal{O}_k^n(H^A)$ will be denoted $\mathrm{Sp}_k^n(A)$. By Theorem 3.8 we have bijections:

$$[\mathrm{Sp}_n(A)] \to [\Delta_A, \mathrm{Sp}_n(\mathbb{C})] = [\Delta_A, \mathrm{Sp}_n(R)],$$

$$[\mathrm{Sp}_k^n(A)] \to [\Delta_A, \mathrm{Sp}_k^n(\mathbb{C})] = [\Delta_A, \mathrm{Sp}_k^n(R)],$$

and

$$[\mathrm{ID}^S(H^A)] \to [\Delta_A, \mathrm{ID}^S(H_n^-)] = [\Delta_A, \mathrm{ID}^S(H_n)].$$

5.5. *Some Special Cases*

The spaces $O_1{}^n(R)$, $U_1{}^n(\mathbb{C})$, and $\mathrm{Sp}_1{}^n(R)$ are particularly interesting. They consist of the sets of unit vectors in R^n, $\mathbb{C}^n$, and H^n, respectively. Hence, $O_1{}^n(R) = S^{n-1}$, $U_1{}^n(\mathbb{C}) = S^{2n-1}$, and $\mathrm{Sp}_1{}^n(R) = S^{4n-1}$.

The corresponding sets $O_1{}^n(A)$, $L_1{}^n(A)$, and $\mathrm{Sp}_1{}^n(A)$ are the sets

$$O_1{}^n(A) = \{\alpha \in A^n : \alpha \cdot \alpha = 1\},$$

$$L_1{}^n(A) = \{\alpha \in A^n : \alpha \text{ is a nonsingular } n\text{-tuple}\}$$

and

$$\mathrm{Sp}_1{}^n(A) = \{\alpha \in H_n{}^A : \alpha' \cdot \alpha = 1\}.$$

We have bijections

$$[O_1{}^n(A)] \to [\Delta_A, S^{n-1}],$$

$$[L_1{}^n(A)] \to [\Delta_A, S^{2n-1}],$$

and

$$[\mathrm{Sp}_1{}^n(A)] \to [\Delta_A, S^{4n-1}].$$

For a topological space X, the set $[X, S^n]$ is called the nth cohomotopy set of X (cf., [20, Chap. VII]).

6. Quadratic Modules

The bijection $[\mathrm{ID}(A \otimes A] \to [\Delta_A, \mathrm{ID}(A)]$ of Theorem 3.8 has a strong implication for the study of finitely generated projective modules over $A \otimes A$. In fact, it implies that there is a bijective correspondence between isomorphism classes of such modules and isomorphism classes of finitely generated projective modules over $C(\Delta_A) \otimes A$ (Theorem 6.8). We shall obtain this as a special case of a more general result that uses the bijection $[\mathrm{ID}((A \otimes A)^+)] \to [\Delta_A, \mathrm{ID}(A^+)]$ for a symmetric algebra A. The appropriate class of modules for this result is the class of finitely generated projective modules with a special kind of bilinear form. These are the quadratic modules (cf., [7]).

Our results on modules lead to isomorphisms between groups from K-theory for $A \otimes A$ and for $C(\Delta_A) \otimes A = C(\Delta_A, A)$.

6.1. *Modules*

Let B be a unital algebra. A right module over B is a vector space M together with a bilinear map $(b, m) \to mb$: $B \times M \to M$ such that

$m(bc) = (mb)c$ and $m1 = m$ for $b, c \in B$ and $m \in M$. Henceforth, the term "module" will mean "right module."

A homomorphism $f: M \to N$ between two modules is a linear map such that $f(mb) = f(m)b$ for $b \in B$, $m \in M$. The vector space of all homomorphisms from M to N will be denoted $\mathrm{Hom}_B(M, N)$ or simply $\mathrm{Hom}(M, N)$ if the algebra B is understood. Under composition, $\mathrm{Hom}(M, M)$ is a unital algebra, which is denoted $\mathrm{End}(M)$.

If $\{M_i\}$ is a family of modules then the vector space direct sum $\bigoplus_i M_i$ is also a module, where the module operation is defined coordinatewise.

The algebra B can be considered a B-module with right multiplication as the module operation. A free module is a module that is isomorphic to $\bigoplus_i B_i$ for some family $\{B_i\}$ of copies of B. A module M is projective if there exists a module N such that $M \oplus N$ is free.

A system of generators for a module M is a set $E \subset M$ such that each $m \in M$ can be written as $\sum e_i b_i$ for finite sets $\{e_i\} \subset E$ and $\{b_i\} \subset B$. A system of generators E is called a basis if each $m \in M$ has a unique representation of this form. A module is free if and only if it has a basis.

A finitely generated module is a module with a finite set of generators. Clearly, in a module that is both free and finitely generated, a basis must be finite. Hence, such a module must be isomorphic to B^n for some n, where B^n denotes the direct sum of n copies of B. Similarly, a module is finitely generated and projective if and only if it is isomorphic to a direct summand of B^n for some n.

The endomorphisms of B^n are easily described. In fact, if $f \in \mathrm{End}(B^n)$ and $e_1, \ldots, e_n$ is the canonical basis for B^n, then there is a unique matrix $(b_{ij}) \in B_n$ defined by $f(e_j) = \sum_i e_i b_{ij}$. Then, if $\alpha = e_1 a_1 + \cdots + e_n a_n \in B^n$; we have

$$f(c) = \sum_j f(e_j)\, a_j = \sum_i e_i \Big(\sum_j b_{ij} a_j\Big).$$

That is, $f(\alpha) = \beta\alpha$, where β is the matrix (b_{ij}), α is written as a column vector (a_j), and $\beta\alpha$ is the usual matrix product. Thus, $\mathrm{End}(B^n)$ may be identified with the matrix algebra B_n.

6.2. *Symmetric Forms*

Let B be a symmetric algebra. If M is a B-module then a symmetric form on M is a bilinear map $(m, n) \to \langle m, n\rangle$: $M \times M \to B$ such that

$$\langle mb, n\rangle = \langle m, n\rangle b$$

and

$$\langle m, n\rangle = \langle n, m\rangle',$$

for $m, n \in M$ and $b \in B$. It follows that

$$\langle m, nb\rangle = b'\langle m, n\rangle.$$

Now, B itself has a natural symmetric form defined by $\langle a, b\rangle = b'a$. Similarly, the canonical symmetric form on B^n is given by

$$\langle \alpha, \beta\rangle = b_1'a_1 + \cdots + b_n'a_n$$

for $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$.

For a module M we denote by M' the module $\mathrm{Hom}(M, A)$, where the module operation is defined by $(fb)(m) = b'f(m)$ for $b \in B$, $f \in M'$, and $m \in M$. Then, a symmetric form $\langle\ ,\ \rangle$ on M defines a module homomorphism $\eta\colon M \to M'$ by

$$\eta(n)(m) = \langle m, n\rangle.$$

The form is called nonsingular if η is an isomorphism. This means simply that each $f \in M'$ has the form $f(m) = \langle m, n\rangle$ for a unique $n \in M$.

It is a simple exercise to show that the canonical symmetric form defined above on B^n is nonsingular.

By a symmetric module, we shall mean a module M together with a symmetric form $\langle\ ,\ \rangle$ on M. A submodule N of a symmetric module M will be called symmetric if $\langle\ ,\ \rangle$ is nonsingular when restricted to N. For any submodule N, we set

$$N^\perp = \{m \in M\colon \langle m, n\rangle = 0, \text{ for all } n \in N\}.$$

LEMMA. *If M is a symmetric module and N a submodule, then N is symmetric if and only if $M = N \oplus N^\perp$.*

Proof. If $M = N \oplus N^\perp$, then every element of N' has a unique extension in M' that vanishes on $N^\perp$. Since $\langle\ ,\ \rangle$ is nonsingular on M, each such element of M' has the form $m \to \langle m, n\rangle$ for a unique $n \in (N^\perp)^\perp$. Since $M = N \oplus N^\perp$, it follows easily that $(N^\perp)^\perp = N$. Hence, each element of N' has the form $n_1 \to \langle n_1, n\rangle$ for a unique $n \in N$.

Conversely, if $\langle\ ,\ \rangle$ is nonsingular on N and $m \in M$, then $n \to \langle n, m\rangle$ determines an element of N' that has the form $n \to \langle n, n_1\rangle$ for a unique $n_1 \in N$. Then, $m - n_1 \in N^\perp$ and $m = n_1 + (m - n_1)$. The uniqueness of n_1 implies that this is the only way to decompose m as an element of N plus an element of $N^\perp$. Hence, $M = N \oplus N^\perp$.

6.3. *Orthogonal Homomorphisms*

If M and N are symmetric modules and $\varphi \in \mathrm{Hom}(M, N)$, then there is an element $\varphi' \in \mathrm{Hom}(N, M)$ defined by the equations

$$\langle \varphi(m), n \rangle = \langle m, \varphi'(n) \rangle$$

for $m \in M$, $n \in N$. That is, $\varphi' = \eta_M^{-1} \varphi^* \eta_N$, where $\eta_M\colon M \to M'$ and $\eta_N\colon N \to N'$ are the isomorphisms determined by the forms and $\varphi^*\colon N' \to M'$ is the dual map: $\varphi^*(f)(m) = f(\varphi(m))$.

We call φ an orthogonal homomorphism if $\varphi'\varphi = 1\colon M \to M$. Note that this means exactly that

$$\langle \varphi(m_1), \varphi(m_2) \rangle = \langle m_1, m_2 \rangle$$

for all $m_1, m_2 \in M$. The set of all orthogonal homomorphisms from M to N will be denoted $O(M, N)$. The orthogonal endomorphisms of M form a group that we denote by $O(M)$.

Note that an element φ of $O(M, N)$ embeds M as a symmetric submodule $\varphi(M)$ of N and that $N = \varphi(M) \oplus \varphi(M)^\perp$ by the previous lemma.

For free modules B^k and B^n, an element $\varphi \in \mathrm{Hom}(B^k, B^n)$ is represented by an $n \times k$ matrix $\beta = (b_{ij})$ over B. It is easy to see that (relative to the canonical forms on B^k and B^n) φ' is represented by the matrix $\beta' = (b'_{ij})^t$. Hence, $O(B^k, B^n)$ is just the space $\mathcal{O}_k{}^n(B)$ defined in Section 3.

6.4. *Quadratic Modules*

A symmetric module that is finitely generated and projective will be called a quadratic module (cf., [7]).

PROPOSITION. *A symmetric module M is quadratic if and only if there is an orthogonal embedding of M into B^n for some n.*

Proof. If $\varphi\colon M \to B^n$ is orthogonal, then $B^n = \varphi(M) \oplus \varphi(M)^\perp$ and hence, $M \simeq \varphi(M)$ is finitely generated and projective.

Conversely, if M is finitely generated and projective, then $M \oplus N \simeq B^n$ for some module N and some n. Let $\Psi\colon M \to B^n$ and

$\theta\colon B^n \to M$ be the corresponding injection and projection. We define a homomorphism $\gamma\colon M \to B^n \oplus B^n$ by

$$\gamma(m) = (2^{1/2} + i(2)^{1/2})^{-1}(\Psi(m) + i\theta'(m), i\Psi(m) + \theta'(m)).$$

Then, since $\theta \circ \Psi = 1$, we have for $m, n \in M$,

$$\begin{aligned}
&\langle \gamma(m), \gamma(n)\rangle \\
&\quad = (1/4i)(\langle \Psi(m) + i\theta'(m), \Psi(n) + i\theta'(n)\rangle + \langle i\Psi(m) + \theta'(m), i\Psi(n) + \theta'(n)\rangle) \\
&\quad = \tfrac{1}{2}(\langle \theta'(m), \Psi(n)\rangle + \langle \Psi(m), \theta'(n)\rangle) \\
&\quad = \tfrac{1}{2}(\langle m, \theta\Psi(n)\rangle + \langle \theta\Psi(m), n\rangle) \\
&\quad = \langle m, n\rangle.
\end{aligned}$$

Hence, γ is an orthogonal embedding of M in B^{2n}.

6.5. *Isomorphism Classes*

We shall denote the set of isomorphism classes of finitely generated projective B-modules by $P(B)$. If B is symmetric then $Q(B)$ will denote the set of orthogonal isomorphism classes of quadratic B-modules.

Each of $P(B)$ and $Q(B)$ is an abelian semigroup under the operation induced by direct sum of modules. The free modules determine a copy of the positive integers in each of $P(B)$ and $Q(B)$.

Let $\varphi\colon B \to C$ be a unital algebra homomorphism and M a B-module. If we consider C to be a left B-module with operation $(b, c) \to \varphi(b)c$, then we can define the tensor product $M \otimes_B C$. This is obtained from the vector space tensor product $M \otimes C$ by factoring out the subspace generated by elements of the form $mb \otimes c - m \otimes \varphi(b)c$. The space $M \otimes_B C$ has a natural right C-module structure defined by $(m \otimes c)d = m \otimes cd$ for $m \in M$, $c, d \in C$.

For the free module B^n, it is easily seen that $B^n \otimes_B C = C^n$. Since the correspondence $M \to M \otimes_B C$ is direct sum preserving, it follows that if M is finitely generated and projective, then $M \otimes_B C$ is also. It follows that $M \to M \otimes_B C$ determines a semigroup homomorphism $\tilde{\varphi}\colon P(B) \to P(C)$. In other words, the correspondence $B \to P(B)$ is a functor from unital algebras to abelian semigroups.

If B and C are symmetric and $\varphi\colon B \to C$ is involution preserving, then a symmetric form $\langle\ ,\ \rangle$ on a B-module M induces one on $M \otimes_B C$. On elementary tensors, this is defined by

$$\langle m \otimes c, n \otimes d\rangle = d'\varphi(\langle m, n\rangle)c.$$

It is not difficult to show that if the original form is nonsingular, the induced form is also. It follows that $M \otimes_B C$ is a quadratic module if M is and hence, that $m \to m \otimes_B C$ induces a semigroup homomorphism $\tilde{\varphi}\colon \underset{\sim}{Q}(B) \to \underset{\sim}{Q}(C)$. Thus, $B \to \underset{\sim}{Q}(B)$ is a functor from symmetric algebras to abelian semigroups.

6.6. *Hyperbolic Modules*

If M is a module over B, then $M' = \mathrm{Hom}(M, B)$ is a module over B^{op} if we define the module operation by $(fb)(c) = bf(c)$. Clearly $(B^n)' = (B^{\mathrm{op}})^n$ and $(M \oplus N)' = M' \oplus N'$. Hence, M' is finitely generated and projective if M is.

For each B-module M, we define an $H(B)$-module $H(M) = M \oplus M'$. Here the module operation is defined by $(m, f)(b, c) = (mb, fc)$ for $(m, f) \in H(M)$ and $(b, c) \in H(B) = B \oplus B^{\mathrm{op}}$. There is a natural symmetric form on $H(M)$ defined by

$$\langle (m, f), (n, g) \rangle = f(n) + g(m).$$

If M is finitely generated and projective, then this form is nonsingular and hence, $H(M)$ is a quadratic $H(B)$-module. The proof of this boils down to the fact that $M'' = M$ if M is finitely generated and projective. This is trivial if $M = B^n$, and follows easily for projective M by using the fact that $M \oplus N = B^n$ for some n and N.

The quadratic module $H(M)$ is called the hyperbolic module for the finitely generated projective module M.

PROPOSITION. *The correspondence $M \to H(M)$ induces a semigroup isomorphism $H\colon P(B) \to \underset{\sim}{Q}(H(B))$.*

Proof. If $\varphi\colon M \to N$ is an isomorphism, then $H(\varphi)(m, f) = (\varphi(m), f \circ \varphi^{-1})$ defines an orthogonal isomorphism from $H(M)$ to $H(N)$. One easily checks that each orthogonal isomorphism $H(M) \to H(N)$ arises in this fashion. Hence, $H\colon P(B) \to \underset{\sim}{Q}(B)$ is well defined and injective. It clearly preserves direct sums and hence, is a semigroup homomorphism.

How suppose M is any quadratic $H(B)$-module. We let $p \in H(B)$ be the idempotent $(1, 0)$ and note that $p' = (0, 1) = 1 - p$. We set $N = Mp$ and $K = Mp'$. Then $M = N \oplus K$. If $m, n \in N$ then

$$\langle m, n \rangle = \langle mp, np \rangle = p'\langle m, n \rangle p = \langle m, n \rangle p'p = 0.$$

Similarly $\langle m, n\rangle = 0$ if $m, n \in K$. It follows that if $m_1 = n_1 + k_1$, $m_2 = n_2 + k_2$ with $n_1, n_2 \in N$ and $k_1, k_2 \in K$, then

$$\langle m_1, m_2\rangle = \langle n_1, k_2\rangle + \langle k_1, n_2\rangle.$$

This implies that the isomorphism $M \to M'$, induced by the form, maps N onto K' and K onto N'. It follows easily that M is isomorphic to $H(N)$. Hence, the map $H: P(B) \to Q(H(B))$ is surjective.

6.7. *Idempotents and Modules*

If M is a direct summand of B^n, then there is an idempotent $p \in \mathrm{End}(B^n)$ with $M = pB^n$. If we identify $\mathrm{End}(B^n)$ with the matrix algebra B_n, then we may regard p as an idempotent in $\mathrm{ID}(B_n)$.

If B is symmetric and $p \in \mathrm{ID}(B_n)$ projects on $M \subset B^n$, then p' is also an idempotent and $\langle p\alpha, \beta\rangle = \langle \alpha, p'\beta\rangle$. It follows that the kernel of p' is exactly $M^\perp$. Hence, $p = p'$ if and only if $B^n = M \oplus M^\perp$ and p is the projection onto M that vanishes on $M^\perp$. Thus, in the symmetric case, there is a one to one correspondence between symmetric submodules of B^n and elements of $\mathrm{ID}(B_n^+)$.

If $m > n$ then an idempotent $p \in \mathrm{ID}(B_n)$ determines one $p \oplus 0 \in \mathrm{ID}(B_m)$. Here we write $B^m = B^n \oplus B^{m-n}$ and define $p \oplus 0$ by $(p \oplus 0)(\alpha, \beta) = (p\alpha, 0)$. Note that if p is symmetric then so is $p \oplus 0$. In this way, we may regard $\mathrm{ID}(B_n)$ and $\mathrm{ID}(B_n^+)$ as subsets of $\mathrm{ID}(B_m)$ and $\mathrm{ID}(B_m^+)$, respectively, whenever $m > n$. If B is a Banach algebra (so that $\mathrm{ID}(B_n)$ and $\mathrm{ID}(B_n^+)$ are topological spaces), then these embeddings are homoemorphisms. We then set

$$\mathrm{ID}(B_\infty) = \operatorname{inj\,lim} \mathrm{ID}(B_n) \qquad \text{and} \qquad \mathrm{ID}(B_\infty^+) = \operatorname{inj\,lim} \mathrm{ID}(B_n^+).$$

PROPOSITION. *If B is a Banach algebra, there is a bijection* $[\mathrm{ID}(B_\infty)] \to P(B)$ *and if B is also symmetric a bijection* $[\mathrm{ID}(B_\infty^+)] \to Q(B)$.

Proof. The first statement follows from the second on passing to the hyperbolic algebra. Hence, we prove the second.

If $p \in \mathrm{ID}(B_n^+) \subset \mathrm{ID}(B_\infty^+)$ we assign to p the class on $Q(B)$ of the module $M = pB^n$. If $m > n$ then $(p \oplus 0)\, B^m$ is clearly isomorphic to M and hence, we have a well defined map $\mathrm{ID}(B_\infty^+) \to Q(B)$. This map is obviously surjective since each quadratic module is an orthogonal direct summand of some B^n. To complete the proof, we must show that two idempotents are in the same component of $\mathrm{ID}(B_\infty^+)$ if and only if they determine the same element of $Q(B)$.

If $p, q \in \mathrm{ID}(B_\infty{}^+)$ are in the same component, then, for large enough n, they may be regarded as elements in the same component of $\mathrm{ID}(B_n{}^+)$. However, by 3.5, this implies that $p = uqu^{-1}$ for some $u \in O(B_n)$. The restriction of u to qB^n is then an orthogonal The restriction of u to qB^n is then an orthogonal isomorphism from qB^n to pB^n. Hence, p and q determine the same element of $\underset{\sim}{O}(B)$.

Conversely, suppose p and q determine the same element of $\underset{\sim}{O}(B)$. Then, for some n, we may regard p and q as elements of $\mathrm{ID}(B_n{}^+)$ with pB^n and qB^n orthogonally isomorphic modules. Let $u\colon qB^n \to pB^n$ be the isomorphism. We then define an orthogonal isomorphism $v\colon B^{2n} \to B^{2n}$ by writing $B^{2n} = pB^n \oplus (1-p)\, B^n \oplus qB^n \oplus (1-q)\, B^n$ and letting v be the map

$$(\alpha, \beta, \gamma, \delta) \to (u^{-1}(\alpha), \beta, u(\gamma), \delta),$$

followed by the map from $B^n \oplus B^n$ to itself that interchanges the two factors. Then, $p \oplus 0 = v(q \oplus 0)\, v^{-1}$ and so the miages of p and q in $\mathrm{ID}(B_{2n}^+)$ are similar via an orthogonal matrix.

To complete the proof, we show that if elements $p, q \in \mathrm{ID}(B_n{}^+)$ are similar via an orthogonal matrix, then their images in $\mathrm{ID}(B_{2n}^+)$ are similar via a matrix in the identity component of $O(B_{2n})$ and hence, are in the same component of $\mathrm{ID}(B_{2n}^+)$. In fact, if $p = vqv^{-1}$ for $v \in O(B_n)$, then $(p \oplus 0) = (v \oplus v^{-1})(q \oplus 0)(v \oplus v^{-1})^{-1}$ in B_{2n}, where

$$v \oplus v^{-1} = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \in O(B_{2n}).$$

Furthermore,

$$t \to w_t = \frac{1}{2}\begin{pmatrix} (1+v) + (1-v)\cos t & (1-v)\sin t \\ (v^{-1}-1)\sin t & (1+v^{-1}) + (1-v^{-1})\cos t \end{pmatrix}$$

is an arc in $O(B_{2n})$ connecting $1 = w_0$ to $v \oplus v^{-1} = w_\pi$.

We are now in a position to give our main application of the results of Section 3 to the study of modules.

6.8. THE MAIN THEOREM. Let A be a unital commutative Banach algebra and Λ a finite-dimensional unital algebra. Then, the Gelfand transform $A \to C(\Delta_A)$ induces a unital algebra homomorphism $A \otimes \Lambda \to C(\Delta_A) \otimes \Lambda = C(\Delta_A, \Lambda)$. If Λ is a symmetric algebra, then this is an involution preserving homomorphism. Hence, there is an induced

THEOREM. *There are natural isomorphisms*

$$\underset{\sim}{Q}(H^A) \to \mathrm{Vect}_H(\Delta_A)$$

$$K\mathrm{Sp}_0(A) \to K\mathrm{Sp}^0(\Delta_A),$$

and

$$K\mathrm{Sp}_{-1}(A) \to K\mathrm{Sp}^{-1}(\Delta_A).$$

If we define B_{Sp} as we did B_U and B_O, then $Z \times B_{\mathrm{Sp}}$ is a classifying space for $K\mathrm{Sp}_0$ and $K\mathrm{Sp}^0$. Of course, Sp is a classifying space for $K\mathrm{Sp}_{-1}$ and $K\mathrm{Sp}^{-1}$.

7.10. *Real K-theory as a Cohomology Theory*

The Bott periodicity theorem in the real case asserts that there are homotopy equivalences (cf., [8]):

$$\begin{aligned} Z \times B_0 &\simeq \Omega(U/O), & Z \times B_{\mathrm{Sp}} &\simeq \Omega(U/\mathrm{Sp}) \\ U/O &\simeq \Omega(\mathrm{Sp}/U), & U/\mathrm{Sp} &\simeq \Omega(O/U) \\ \mathrm{Sp}/U &\simeq \Omega(\mathrm{Sp}), & O/U &\simeq \Omega(0) \\ \mathrm{Sp} &\simeq \Omega(Z \times B_{\mathrm{Sp}}), & 0 &\simeq \Omega(Z \times B_0). \end{aligned}$$

This leads to a cohomology theory of period eight in which the above spaces are classifying spaces for the eight functors involved. We have related four of these functors to the structure of Banach algebras. They are:

$$\begin{aligned} K_R{}^0 &= KO^0, & K_R{}^4 &= K\mathrm{Sp}^0 \\ K_R{}^3 &= K\mathrm{Sp}^{-1}, & K_R{}^7 &= KO^{-1}. \end{aligned}$$

The remaining functors of real K-theory have classifying spaces U/O, Sp/U, U/Sp, and O/U (for appropriate injections of the groups O, U, Sp into one another). Each of these spaces has the same homotopy type as a space which is a limit of complex homogeneous spaces. Hence, the results of Section 2 should yield direct description of the remaining functors of real K-theory in Banach algebra terms. However, we will not attempt to give such descriptions here.

7.11. *Cech Cohomology*

Cech cohomology is the unique cohomology theory (in the sense we are using the term) that satisfies the dimension axiom. For a given group homomorphism $s \to [s]\colon S \to U(S)$, where $[s] \in U(S)$ is the class containing all elements $(s + t, t)$. The group $U(S)$ and homomorphism $S \to U(S)$ are determined up to isomorphism by the following universal condition:

Each homomorphism $S \to G$ of S into a group G factors uniquely as the composition

$$S \to U(S) \to G$$

for a group homomorphism $U(S) \to G$.

A subsemigroup J of S is called cofinal if for each $s \in S$, there is a $t \in S$ such that $s + t \in J$. If J is cofinal, then it is easily seen that each element of $U(S)$ has the form $[s] - [j]$ for $s \in S, j \in J$ and that for s, $t \in S$, $[s] = [t]$ if and only if $s + j = t + j$ for some $j \in J$.

7.2. *The Groups* $K_0(B)$ *and* $KO_0(B)$

If B is a unital algebra, then $K_0(B)$ will denote the universal group of the semigroup $P(B)$ of Section 6. This is the Grothendieck group for B (cf., [6]). If B is symmetric then $KO_0(B)$ will denote the universal group of $\underset{\sim}{Q}(B)$ (in [7], this is denoted $KU(B)$).

Each of $P(B)$ and $\underset{\sim}{Q}(B)$ has a copy of the integers (determined by the free modules) as a cofinal subsemigroup. It follows that each element of $K_0(B)$ (resp. $KO_0(B)$) has the form $[M] - n$, where M is a finitely generated projective (resp. quadratic) B-module, and n represents the class of a free module B^n. Furthermore, $[M] = [N]$ for finitely generated projective (resp. quadratic) modules M and N if and only if $M \oplus B^k$ is isomorphic (resp. orthogonally isomorphic) to $N \oplus B^k$ for some integer k. In this case, we say that M and N are stably isomorphic.

7.3. *The Groups* $K_{-1}(B)$ *and* $KO_{-1}(B)$

Let B be a unital Banach algebra. For $m > n$ and $u \in B_n^{-1}$ we define $u \oplus 1 \in B_m^{-1}$ by

$$u \oplus 1 = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, $u \to u \oplus 1\colon B_n^{-1} \to B_m^{-1}$ embeds B_n^{-1} as a subgroup of B_m^{-1}. We set

$$K_{-1}(B) = \lim_{n} [B_n^{-1}].$$

Similarly, in the symmetric case, $O(B_n)$ is embedded as a subgroup of $O(B_m)$ for $m > n$ and we define

$$KO_{-1}(B) = \lim_{n} [O(B_n)].$$

Each of $K_{-1}(B)$ and $KO_{-1}(B)$ is an abelian group. In fact, if $u, v \in B_n^{-1}$ (resp. $O(B_n)$), then

$$t \to \begin{pmatrix} u \cos^2 t + v \sin^2 t & (u - v) \sin t \cos t \\ (u - v) \sin t \cos t & u \sin^2 t + v \cos^2 t \end{pmatrix}$$

is an arc in B_{2n}^{-1} (resp. $O(B_{2n})$), which is $u \oplus v$ when $t = 0$ and $v \oplus u$ when $t = \pi/2$. Hence, $u \oplus v$ and $v \oplus u$ determine the same class in $[B_{2n}^{-1}]$ (resp. $[O(B_{2n})]$). It follows that

$$\begin{aligned} uv \oplus 1 = (u \oplus 1)(v \oplus 1) &\sim (u \oplus 1)(1 \oplus v) = u \oplus v \\ &\sim v \oplus u = (v \oplus 1)(1 \oplus u) \\ &\sim (v \oplus 1)(u \oplus 1) = vu \otimes 1, \end{aligned}$$

where $a \sim b$ means a and b are in the same component of B_{2n}^{-1} (resp. $O(B_{2n})$). We conclude that the operation defined in $K_{-1}(B)$ (resp. $KO_{-1}(B)$) by matrix multiplication is abelian and could also have been defined by direct sum.

The group $K_{-1}(B)$ is closely related to the whitehead group of algebraic K-theory (cf., [6]) and, in fact, can be made to play the same role in a K-theory for Banach algebras (cf., [22, 32]). If B is commutative, then $K_{-1}(B)$ is actually a direct summand of the whitehead group, where the complementary summand is a copy of $[B^{-1}] = H^1(\Delta_B, Z)$ (cf., [6, Chap. XIV]).

As is true for the whitehead group, K_{-1} is related to K_0 in the following way: If $I \subset B$ is a closed ideal, then there is a group $\tilde{K}_0(I)$ and an exact sequence

$$K_{-1}(B) \to K_{-1}(B/I) \to \tilde{K}_0(I) \to K_0(B) \to K_0(B/I)$$

(cf., [6, 32]). There is a similar exact sequence relating KO_{-1} and KO_0.

Our concern here is not with the internal machinery of K-theory itself, but rather with results of the following sort:

7.4. THEOREM. *Let A be a finite-dimensional unital algebra and A a* *unital commutative Banach algebra. Then, the Gelfand transform induces isomorphisms*

$$K_0(A \otimes \Lambda) \to K_0(C(\Delta_A, \Lambda)),$$
$$K_{-1}(A \otimes \Lambda) \to K_{-1}(C(\Delta_A, \Lambda)),$$

and in the case where Λ is symmetric,

$$KO_0(A \otimes \Lambda) \to KO_0(C(\Delta_A, \Lambda)),$$
$$KO_{-1}(A \otimes \Lambda) \to KO_{-1}(C(\Delta_A, \Lambda)).$$

Proof. This is a direct consequence of Theorem 6.8 and Theorem 3.8.

7.5. *Complex K-theory*

We now specialize to the case $\Lambda = \mathbb{C}$ and discuss $P(A)$, $K_0(A)$, and $K_{-1}(A)$ for a unital commutative Banach algebra A.

In case $A = C(X)$ for a compact Hausdorff space X, it is well known that $P(A)$ is isomorphic to $\mathrm{Vect}_{\mathbb{C}}(X)$, the semigroup (under direct sum) of isomorphism classes of complex vector bundles on X (cf., [4, 1.4]). The isomorphism is easily described. If M is a module direct summand of $C(X)^n$ representing an element of $P(C(X))$ and if $p \in \mathrm{ID}(C(X)^n)$ projects on M, then for each $x \in X$, $p(x)$ projects on a vector subspace V_x of $\mathbb{C}^n$. The subspace $\{(x, v) \in X \times \mathbb{C}^n: v \in V_x\}$ is a vector bundle over X with fiber V_x at x, and in fact, is a sub-bundle of the trivial bundle $X \times \mathbb{C}^n$. The class of this bundle in $\mathrm{Vect}_{\mathbb{C}}(X)$ is the image of the class of M in $P(C(X))$.

The universal group of the semigroup $\mathrm{Vect}_{\mathbb{C}}(X)$ is the group $K^0(X)$ of complex topological K-theory (cf., [4]). Similarly, the group $K_{-1}(C(X)) = \lim_n [X, GL_n(\mathbb{C})]$ is the group $K^{-1}(X)$ of complex topological K-theory.

In view of Theorems 6.8 and 7.4, we have the following result of Novodvorskii [24] (also see [15, 32]):

THEOREM. *If A is a unital commutative Banach algebra, then the Gelfand transform induces isomorphisms*

$$P(A) \to \mathrm{Vect}_{\mathbb{C}}(\Delta_A),$$
$$K_0(A) \to K^0(\Delta_A),$$

and

$$K_{-1}(A) \to K^{-1}(\Delta_A).$$

The Bott periodicity theorem in the complex case allows one to construct a cohomology theory $\{K^p\}$ in which $K^p = K^0$ for p even and $K^p = K^{-1}$ for p odd (cf., [4, 13, 32]). The above theorem gives the exact relationship between the groups of this cohomology theory for Δ_A and the structure of A.

7.6. *Classifying Spaces for K-theory*

By Proposition 6.7, $P(A) \simeq [\mathrm{ID}(A_\infty)] = \lim_n [\mathrm{ID}(A_n)]$. It follows from 5.3 that

$$\mathrm{ID}((C_\infty)^h) = \lim_n \mathrm{ID}((C_n)^h)$$

is a classifying space for the functor $A \to P(A)$. An element of $\mathrm{ID}((\mathbb{C}_\infty)^h)$ can be thought of as the unique Hermitian idempotent associated with a finite-dimensional subspace of $\mathbb{C}^\infty$, where $\mathbb{C}^\infty$ is the space of finitely nonzero sequences $\{\lambda_1, \lambda_2, \ldots\}$ of elements of $\mathbb{C}$. Hence, we may identify $\mathrm{ID}((\mathbb{C}_\infty)^h)$ with the discrete union of the spaces B_{U_k}, where B_{U_k} is the space of k-dimensional subspaces of $\mathbb{C}^\infty$. The space B_{U_k} is called the classifying space for the unitary group U_k (cf., [21, Chap. 7]).

The isomorphism $[\mathrm{ID}(A_\infty)] \simeq [\Delta_A, \bigcup_k B_{U_k}]$ maps $[\mathrm{ID}_k(A_\infty)]$ onto $[\Delta_A, B_{U_k}]$ for each k, where $\mathrm{ID}_k(A_\infty)$ is the space of idempotents of constant rank k in $\mathrm{ID}(A_\infty) = \lim \mathrm{ID}(A_n)$ (cf., 5.3). We say a finitely generated projective A-module M has constant rank k if M corresponds to an idempotent of constant rank k. The set of isomorphism classes of such modules forms a subset $P^k(A)$ of $P(A)$. If $\mathrm{Vect}_C^k(X)$ denotes the set of isomorphism classes of vector bundles on X with fibers of constant dimension k, then we have

$$P^k(A) \simeq [\Delta_A, B_{U_k}] \simeq \mathrm{Vect}_C^k(\Delta_A).$$

The classifying space $\mathrm{ID}((\mathbb{C}_\infty)^h) = \bigcup_k B_{U_k}$ for $A \to P(A)$ has an H-space structure (cf., 1.2), which induces the semigroup operation on $P(A)$. This is described as follows: Let $\varphi\colon \mathbb{C}^\infty \oplus \mathbb{C}^\infty \to \mathbb{C}^\infty$ be any vector space isomorphism. Then, if V and W are subspaces of $\mathbb{C}^\infty$ of dimensions k and l respectively, $\varphi(V \oplus W)$ is a subspace of dimension $k + l$. This yields a map $\tilde{\varphi}\colon B_{U_k} \times B_{U_l} \to B_{U_{k+l}}$ for each pair k, l. The resulting operation on $\bigcup_k B_{U_k}$ makes it into an H-semigroup. That this operation induces the direct sum operation on $P(A) \simeq [\Delta_A, \bigcup_k B_{U_k}]$ is reasonably clear.

To find a classifying space for $A \to K_0(A)$, we proceed as follows: Let E denote the span of the first basis vector of $\mathbb{C}^\infty$. Then,

$V \to \tilde{\varphi}(V, E) = \varphi(V \oplus E)$ maps B_{U_k} homeomorphically into $B_{U_{k+1}}$ for each k. The space

$$B_U = \lim_k B_{U_k}$$

has the property that $[\Delta_A, B_U]$ is obtained from the semigroup $[\Delta_A, \bigcup_k B_{U_k}] = P(A)$ by identifying elements x, y whenever $x + k = y + l$ for positive integers (classes of free modules) k and l. The resulting set is a factor semigroup of $P(A)$ and, in fact, a group that is isomorphic to $K_0(A)/Z$. To obtain a classifying space for K_0 (or K^0), we simply take $Z \times B_U$ (cf., [13, Chap. III]).

The space B_U is an H-group and is called the classifying space of the infinite unitary group.

An alternative classifying space for K_0 is the space of all Fredholm operators on Hilbert space or, equivalently, the invertible group of the Calkin algebra (cf., [5, Theorem 2.3]).

The infinite unitary group

$$U = \lim_n U_n$$

is clearly a classifying space for K_{-1} (or K^{-1}) (by 5.3). The fact that K^0 and K^{-1} determine a cohomology theory of period 2 is due to the fact that there are homotopy equivalences

$$\Omega U \simeq Z \times B_U, \qquad \Omega(Z \times B_U) \simeq U,$$

where ΩX is the loop space of X (cf., [13, Chap. III]).

7.7. *The Picard Group*

Since A is commutative, the tensor product $M \otimes_A N$ of two A-modules is also an A-module. Since tensor product distributes over direct sum and since $A^n \otimes A^m \simeq A^{nm}$, we conclude that the tensor product of two finitely generated projective modules is another one. Thus, tensor product induces an abelian multiplication on $P(A)$ that distributes over addition. With this additional operation, $P(A)$ is an abelian semiring. The rank one free module A is an identity for this semiring.

The Picard group $\operatorname{Pic}(A)$ is the group of invertible elements of $P(A)$ (cf., [6, II. Sect. 5]).

If X is a compact Hausdorff space, then it is easy to see that $\operatorname{Vect}_{\mathbb{C}}(X)$ is also a semiring with the multiplication defined by vector bundle tensor product (cf., [4]). The invertible group of $\mathrm{Vect}_{\mathbb{C}}(X)$ is the group (under tensor product) of isomorphism classes of line bundles (one-dimensional vector bundles). Now, the map $P(A) \to \mathrm{Vect}_{\mathbb{C}}(\Delta_A)$ is an isomorphism of semirings as well as an isomorphism of semigroups. Furthermore, the group of line bundles in $\mathrm{Vect}_{\mathbb{C}}(\Delta_A)$ is the image of $P_1(A)$, the set in $P(A)$ determined by modules of constant rank 1. Hence, $P_1(A) = \mathrm{Pic}(A)$.

By 7.6, B_{U_1} is a classifying space for $P_1(A)$. Furthermore, B_{U_1} is just the infinite complex projective plane $P^\infty(\mathbb{C})$. It is well known to topologists that this is an H-group that is a classifying space for the group of isomorphism classes of line bundles (cf., [21]), as well as the second Cech group $H^2(\ ,Z)$ (cf., [31, Chap. 8, Sect. 1]). Hence, we have the following theorem of Forster [15]:

THEOREM. *There is a natural isomorphism* $\mathrm{Pic}(A) \simeq H^2(\Delta_A, Z)$.

7.8. *Real K-theory*

While the K-theory of complex vector bundles leads to a cohomology theory of period 2, the K-theory of real vector bundles leads to a cohomology theory of period 8. To relate the functors of this theory to the structure of Banach algebras requires the machinery of symmetric algebras and quadratic modules discussed earlier.

We consider the symmetric algebra $A \otimes \Lambda$ in the cases where $\Lambda = \mathbb{C}$ (with trivial involution) and where $\Lambda = H^{\mathbb{C}}$, the algebra of complexified quaternions (cf., Sect. 5).

In the first case, $A \otimes \mathbb{C} = A$, where A has the trivial involution. The canonical symmetric form on A^n is given by

$$\langle \alpha, \beta \rangle = \alpha \cdot \beta = a_1 b_1 + \cdots + a_n b_n ,$$

for $\alpha = (a_1, \ldots, a_n)$, $\beta = (b_1, \ldots, b_n)$. The quadratic A-modules are the symmetric modules that are orthogonally isomorphic to symmetric submodules of A^n for some n. The projections $p \in A_n$ corresponding to symmetric submodules are those for which $p^t = p$. In view of 6.7 and 5.1, we have isomorphisms

$$Q(A) \to [\mathrm{ID}(A_\infty{}^+)] \to [\Delta_A, \mathrm{ID}(\mathbb{C}_\infty{}^+)] \to [I_A, \mathrm{ID}(R_\infty{}^+)].$$

Now, $\mathrm{ID}(R_\infty{}^+)$ may be identified with the space of finite-dimensional subspaces of $R^\infty = R \oplus R \oplus \cdots$. As in the complex case, a continuous function on Δ_A with values in this space determines a vector bundle on Δ_A (a real vector bundle in this case). In fact, $\mathrm{ID}(R_\infty{}^+)$ is a classifying space for the functor Vect_R, where $\mathrm{Vect}_R(X)$ is the semigroup of isomorphism classes of real vector bundles (cf., [21]). It follows that we have isomorphisms $Q(A) \to \mathrm{Vect}_R(\Delta_A)$, and on passing to the corresponding universal groups, $KO_0(A) \to KO^0(\Delta_A)$. Here $KO^0(\Delta_A)$ is the Grothendieck group for the category of real vector bundles on Δ_A.

The orthogonal group $O(A_n)$ is the group $O_n(A)$ of matrices $u \in A_n$ with $u^t = u^{-1}$. By 5.1, we have $[O_n(A)] \simeq [\Delta_A, O_n(R)]$. On passing to the direct limit over n we obtain an isomorphisms $KO_{-1}(A) \simeq KO^{-1}(\Delta_A) = [\Delta_A, O]$, where O is the infinite orthogonal group $\lim_n O_n(R)$. Summarizing, we have:

THEOREM. *There are natural isomorphisms*

$$Q(A) \to \mathrm{Vect}_R(\Delta_A),$$

$$KO_0(A) \to KO^0(\Delta_A),$$

$$KO_{-1}(A) \to KO^{-1}(\Delta_A).$$

The classifying space $\mathrm{ID}(R_\infty{}^+)$ breaks up as $\bigcup_k B_{O_k}$, where B_{O_k} is the space of k-dimensional subspaces of R^∞. There are natural maps $B_{O_k} \to B_{O_{k+1}}$ for each k and if $B_O = \lim B_{O_k}$, then $Z \times B_O$ is a classifying space for KO_0 (and KO^0) (cf., [21]). We have already shown that the infinite orthogonal group O is a classifying space for KO_{-1} (and KO^{-1}).

7.9. *The Symplectic Case*

If we choose A to be the symmetric algebra $H^{\mathbf{C}}$, then $A \otimes A = H^A$ (cf., Sect. 5). By Propositions 6.7 and 5.4 there are isomorphisms

$$Q(H^A) \to [\mathrm{ID}((H^A)_\infty^+)] \to [\Delta_A, \mathrm{ID}(H_\infty{}^+)].$$

The space $\mathrm{ID}(H_\infty{}^+)$ can be considered the spaces of all finite-dimensional H-linear subspaces of $H^\infty = H \oplus H \oplus \cdots$. As before, $[\Delta_A, \mathrm{ID}(H_\infty{}^+)]$ may be identified with the semgiroup $\mathrm{Vect}_H(\Delta_A)$ of (H-linear) isomorphism classes of H-vector bundles (i.e., symplectic bundles) (cf., [21]).

We denote the universal group $KO_0(H^A)$ of $Q(H^A)$ by $K\mathrm{Sp}_0(A)$ and the universal group of $\mathrm{Vect}_H(X)$ by $K\mathrm{Sp}^0(A)$. Similarly, we set $K\mathrm{Sp}_{-1}(A) = KO_{-1}(H^A) = \lim[\mathrm{Sp}_n(A)]$ and $K\mathrm{Sp}^{-1}(X) = [X, \mathrm{Sp}]$, where $\mathrm{Sp} = \lim \mathrm{Sp}_n(R)$. By the above and 5.4 we have:

THEOREM. *There are natural isomorphisms*

$$Q(H^A) \to \mathrm{Vect}_H(\Delta_A)$$

$$K\mathrm{Sp}_0(A) \to K\mathrm{Sp}^0(\Delta_A),$$

and

$$K\mathrm{Sp}_{-1}(A) \to K\mathrm{Sp}^{-1}(\Delta_A).$$

If we define B_{Sp} as we did B_U and B_O, then $Z \times B_{\mathrm{Sp}}$ is a classifying space for $K\mathrm{Sp}_0$ and $K\mathrm{Sp}^0$. Of course, Sp is a classifying space for $K\mathrm{Sp}_{-1}$ and $K\mathrm{Sp}^{-1}$.

7.10. *Real K-theory as a Cohomology Theory*

The Bott periodicity theorem in the real case asserts that there are homotopy equivalences (cf., [8]):

$$\begin{aligned} Z \times B_0 &\simeq \Omega(U/O), & Z \times B_{\mathrm{Sp}} &\simeq \Omega(U/\mathrm{Sp}) \\ U/O &\simeq \Omega(\mathrm{Sp}/U), & U/\mathrm{Sp} &\simeq \Omega(O/U) \\ \mathrm{Sp}/U &\simeq \Omega(\mathrm{Sp}), & O/U &\simeq \Omega(0) \\ \mathrm{Sp} &\simeq \Omega(Z \times B_{\mathrm{Sp}}), & 0 &\simeq \Omega(Z \times B_0). \end{aligned}$$

This leads to a cohomology theory of period eight in which the above spaces are classifying spaces for the eight functors involved. We have related four of these functors to the structure of Banach algebras. They are:

$$K_R^0 = KO^0, \qquad K_R^4 = K\mathrm{Sp}^0$$

$$K_R^3 = K\mathrm{Sp}^{-1}, \qquad K_R^7 = KO^{-1}.$$

The remaining functors of real K-theory have classifying spaces U/O, Sp/U, U/Sp, and O/U (for appropriate injections of the groups O, U, Sp into one another). Each of these spaces has the same homotopy type as a space which is a limit of complex homogeneous spaces. Hence, the results of Section 2 should yield direct description of the remaining functors of real K-theory in Banach algebra terms. However, we will not attempt to give such descriptions here.

7.11. *Cech Cohomology*

Cech cohomology is the unique cohomology theory (in the sense we are using the term) that satisfies the dimension axiom. For a given coefficient group π, the functors $H^n(\ ,\pi)$ of Cech cohomology theory are homotopy functors with classifying spaces $K(\pi, n)$ that are H-groups. These are the Eilenberg–McLane spaces (cf., [31, Chap. 8, Sect. 1]).

In the case of integer coefficients, the first three Eilenberg–McLane spaces are

$$K(Z, 0) = Z, \qquad K(Z, 1) = S^1, \qquad K(Z, 2) = P^\infty(\mathbb{C}).$$

These are the standard descriptions. However, we could also choose Z, $\mathbb{C}^{-1} = \mathbb{C}/\{0\}$, and $\mathrm{ID}_1(\mathbb{C}_\infty)$, since each of these has the same homotopy type as the corresponding space above. Each of the latter spaces is a limit of complex homogeneous spaces and hence, is amenable to the machinery of Section 2. It is for this reason, that we have the isomorphisms

$$\begin{aligned} H^0(\Delta_A, Z) &\simeq (\text{group generated by } \mathrm{ID}(A)),\\ H^1(\Delta_A, Z) &\simeq A^{-1}/\exp(A),\\ H^2(\Delta_A, Z) &\simeq \mathrm{Pic}(A), \end{aligned}$$

given by the Shilov [29], Arens–Royden [1, 26], and Forster [15] theorems.

The problem of obtaining similar characterizations of the higher Cech groups is as yet unsolved. There are indications that the techniques we have been employing here may fail for the higher Cech groups. That is, there are indications that it may be impossible to approximate $K(Z, n)$ by complex homogeneous spaces when $n > 2$.

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